

A Graph Theoretical Approach to Network Encoding Complexity

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Abstract

Consider an acyclic directed network G with sources S_1, S_2, \dots, S_l and distinct sinks R_1, R_2, \dots, R_l . For $i = 1, 2, \dots, l$, let c_i denote the min-cut between S_i and R_i . Then, by Menger's theorem, there exists a group of c_i edge-disjoint paths from S_i to R_i , which will be referred to as a group of Menger's paths from S_i to R_i in this paper. Although within the same group they are edge-disjoint, the Menger's paths from different groups may have to merge with each other. It is known that by choosing Menger's paths appropriately, the number of mergings among different groups of Menger's paths is always bounded by a constant, which is independent of the size and the topology of G . The tightest such constant for the all the above-mentioned networks is denoted by $\mathcal{M}(c_1, c_2, \dots, c_l)$ when all S_i 's are distinct, and by $\mathcal{M}^*(c_1, c_2, \dots, c_l)$ when all S_i 's are in fact identical. It turns out that \mathcal{M} and \mathcal{M}^* are closely related to the network encoding complexity for a variety of networks, such as multicast networks, two-way networks and networks with multiple sessions of unicast. Using this connection, we compute in this paper some exact values and bounds in network encoding complexity using a graph theoretical approach.

1 Introduction and Notations

Let $G(V, E)$ denote an acyclic directed graph, where V denotes the set of all the vertices (or points) in G and E denotes the set of all the edges in G . In this paper, a *path* in G is treated as a set of concatenated edges. For k paths $\beta_1, \beta_2, \dots, \beta_k$ in $G(V, E)$, we say these paths *merge* [5] at an edge $e \in E$ if

1. $e \in \bigcap_{i=1}^k \beta_i$,
2. there are at least two distinct edges $f, g \in E$ such that f, g are immediately ahead of e on some β_i, β_j , $i \neq j$, respectively.

We call the maximal subpath that starts with e and that is shared by all β_i 's (i.e., e together with the subsequent concatenated edges shared by all β_i 's until some β_i branches off) *merged subpath* (or simply *merging*) by all β_i 's at e ; see Figure 1 for a quick example.

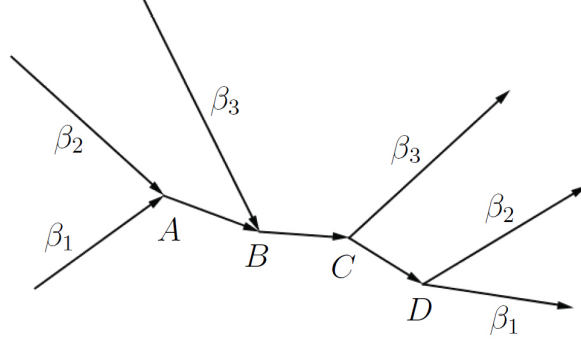


Figure 1: Paths β_1, β_2 merge at edge $A \rightarrow B$ and at merged subpath (or merging) $A \rightarrow B \rightarrow C \rightarrow D$, and paths $\beta_1, \beta_2, \beta_3$ merge at edge $B \rightarrow C$ and at merged subpath (or merging) $B \rightarrow C$.

For any two vertices $u, v \in V$, we call any set consisting of the maximum number of pairwise edge-disjoint directed paths from u to v a set of *Menger's paths* from u to v . By Menger's theorem [8], the cardinality of Menger's paths from u to v is equal to the min-cut between u and v . Here, we remark that the Ford-Fulkerson algorithm [3] can find the min-cut and a set of Menger's paths from u to v in polynomial time.

Assume that $G(V, E)$ has l sources S_1, S_2, \dots, S_l and l distinct sinks R_1, R_2, \dots, R_l . For $i = 1, 2, \dots, l$, let c_i denote the min-cut between S_i and R_i , and let $\alpha_i = \{\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,c_i}\}$ denote a set of Menger's paths from S_i to R_i . We are interested in the number of mergings among paths from different α_i 's, denoted by $|G|_{\mathcal{M}}(\alpha_1, \alpha_2, \dots, \alpha_l)$. In this paper we will count the number of mergings **without** multiplicity: all the mergings at the same edge e will be counted as one merging at e . The motivation for such consideration is more or less obvious in transportation networks: mergings among different groups of transportation paths can cause congestions, which may either decrease the whole network throughput or incur unnecessary cost. The connection between the number of mergings and the encoding complexity in computer networks, however, is a bit more subtle, which can be best illustrated by the following three examples in network coding theory (for a brief introduction to this theory, see [14]).

The first example is the famous “butterfly network” [7]. As depicted in Figure 2(a), for the purpose of transmitting messages a, b simultaneously from the sender S to the receivers R_1, R_2 , network encoding has to be done at node C . Another way to interpret the necessity of network coding at C (for the simultaneous transmission to R_1 and R_2) is as follows: If the transmission to R_2 is ignored, Menger's paths $S \rightarrow A \rightarrow R_1$ and $S \rightarrow B \rightarrow C \rightarrow D \rightarrow R_1$ can be used to transmit messages a, b from S to R_1 ; if the transmission to R_1 is ignored, Menger's paths $S \rightarrow A \rightarrow C \rightarrow D \rightarrow R_2$ and $S \rightarrow B \rightarrow R_2$ can be used to transmit messages a, b from S to R_2 . For the simultaneous transmission to R_1 and R_2 , merging by these two groups of Menger's paths at $C \rightarrow D$ becomes a “bottle neck”, therefore network coding at C is required to avoid the possible congestions.

The second example is a variant of the classical butterfly network (see Example 17.2 of [12]; cf. the two-way channel in Page 519 of [2]) with two senders and two receivers, where the sender S_1 is attached to the receiver R_2 to form a group and the sender S_2 is attached to the receiver R_1 to form the other group. As depicted in Figure 2(b), the two groups wish to

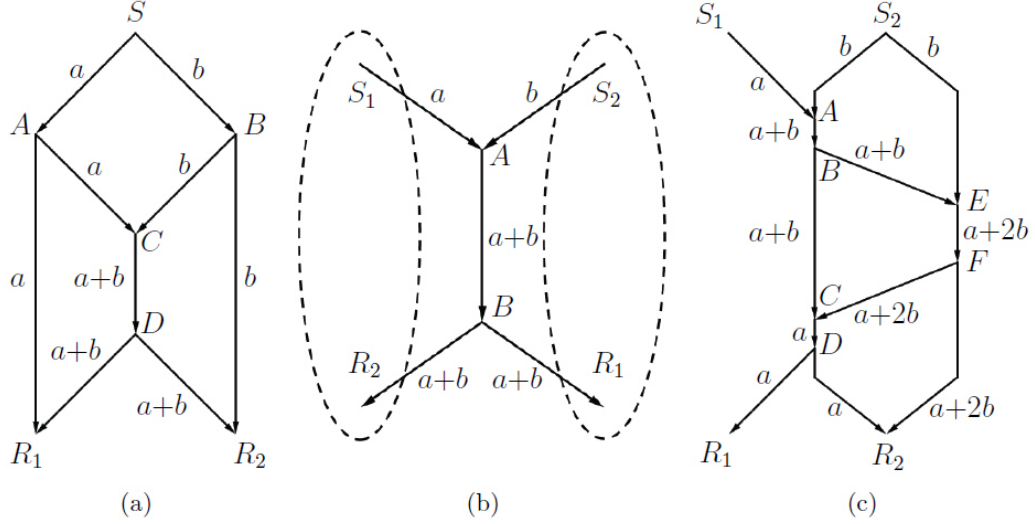


Figure 2: (a) Network coding on the butterfly network (b) Network coding on a variant of the butterfly network (c) Network coding on two sessions of unicast

exchange messages a and b through the network. Similarly as in the first example, the edge $A \rightarrow B$ is where the Menger's paths $S_1 \rightarrow A \rightarrow B \rightarrow R_1$ and $S_2 \rightarrow A \rightarrow B \rightarrow R_2$ merge with each other, which is a bottle neck for the simultaneous transmission of messages a, b . The simultaneous transmission is achievable if upon receiving the messages a and b , network encoding is performed at the node A and the newly derived message $a + b$ is sent over the channel AB .

The third example is concerned with two sessions of unicast in a network [9]. As shown in Figure 2(c), the sender S_1 is to transmit message a to the receiver R_1 using path $S_1 \rightarrow A \rightarrow B \rightarrow E \rightarrow F \rightarrow C \rightarrow D \rightarrow R_1$. And the sender S_2 is to transmit message b to the receiver R_2 using two Menger's paths $S_2 \rightarrow A \rightarrow B \rightarrow C \rightarrow D \rightarrow R_2$ and $S_2 \rightarrow E \rightarrow F \rightarrow R_2$. Since mergings $A \rightarrow B$, $C \rightarrow D$ and $E \rightarrow F$ become bottle necks for the simultaneous transmission of messages a and b , network coding at these bottle necks, as shown in Figure 2(c), is performed to ensure the simultaneous message transmission.

Generally speaking, for a network with multiple groups of Menger's paths, each of which is used to transmit a set of messages to a particular sink, network encoding is needed at mergings by different groups of Menger's paths. As a result, the number of mergings is the number of network encoding operations required in the network. So, we are interested in the number of mergings among different groups of Menger's paths in such networks.

For the case when all sources in G are in fact identical, $M^*(G)$ is defined as the minimum of $|G|_{\mathcal{M}}(\alpha_1, \alpha_2, \dots, \alpha_l)$ over all possible Menger's path sets α_i 's, $i = 1, 2, \dots, l$, and $\mathcal{M}^*(c_1, c_2, \dots, c_l)$ is defined as the supremum of $M^*(G)$ over all possible choices of such G . It is clear that $M^*(G)$ is the least number of network encoding operations required for a given G , and $\mathcal{M}^*(c_1, c_2, \dots, c_l)$ is the largest such number among all such G (with the min-cut between the i -th pair of source and sink being c_i). As for \mathcal{M}^* , the authors of [4] used the idea of "subtree decomposition" to first prove that

$$\mathcal{M}^*(\underbrace{2, 2, \dots, 2}_l) = l - 1.$$

Although their idea seems to be difficult to generalize to other parameters, it does allow us to gain deeper understanding about the topological structure of the graphs achieving $l - 1$ mergings for this special case. It was first shown in [6] that $\mathcal{M}^*(c_1, c_2)$ is finite for all c_1, c_2 (see Theorem 22 in [6]), and subsequently $\mathcal{M}^*(c_1, c_2, \dots, c_l)$ is finite for all c_1, c_2, \dots, c_l .

For the case when all sources in G are distinct, $M(G)$ is defined as the minimum of $|G|_{\mathcal{M}}(\alpha_1, \alpha_2, \dots, \alpha_l)$ over all possible Menger's path sets α_i 's, $i = 1, 2, \dots, l$, and $\mathcal{M}(c_1, c_2, \dots, c_l)$ is defined as the supremum of $M(G)$ over all possible choices of such G . Again, the encoding idea for the second example can be easily generalized to networks, where each receiver is attached to all senders except its associated one. It is clear that the number of mergings is a tight upper bound for the number of network encoding operations required. For networks with several unicast sessions, in [9], an upper bound for the encoding complexity of a network with two unicast sessions was given, as a result of a more general treatment (to networks with two multicast sessions) by the authors. It is easy to see that for networks with multiple unicast sessions (straightforward generalizations of the third example), \mathcal{M} with appropriate parameters can serve as an upper bound on network encoding complexity. It was first conjectured that $\mathcal{M}(c_1, c_2, \dots, c_l)$ is finite in [10]. More specifically the authors proved that (see Lemma 10 in [10]) if $\mathcal{M}(c_1, c_2)$ is finite for all c_1, c_2 , then $\mathcal{M}(c_1, c_2, \dots, c_l)$ is finite as well. Here, we remark that we have rephrased the work in [4, 6, 10], since all of them are done using very different languages from ours.

In [5], we have shown that for any c_1, c_2, \dots, c_l , $\mathcal{M}^*(c_1, c_2, \dots, c_l)$, $\mathcal{M}(c_1, c_2, \dots, c_l)$ are both finite, and we further studied the behaviors of $\mathcal{M}^*, \mathcal{M}$ as functions of the min-cuts. In this paper, further continuing the work in [5], we compute exact values of and give tighter bounds on \mathcal{M}^* and \mathcal{M} for certain parameters.

For a path β in G , let $h(\beta), t(\beta)$ denote *head* (or *starting point*) and *tail* (or *ending point*) of path β , respectively; let $\beta[u, v]$ denote the subpath of β with the starting point u and the ending point v . For two distinct paths ξ, η in G , we say ξ is *smaller* than η (or, η is *larger* than ξ) if there is a directed path from $t(\xi)$ to $h(\eta)$; if ξ, η and the connecting path from $t(\xi)$ to $h(\eta)$ are subpaths of path β , we say ξ is *smaller* than η on β . Note that this definition also applies to the case when paths degenerate to vertices/edges; in other words, in the definition, ξ, η or the connecting path from $t(\xi)$ to $h(\eta)$ can be vertices/edges in G , which can be viewed as degenerated paths. If $t(\xi) = h(\eta)$, we use $\xi \circ \eta$ to denote the path obtained by concatenating ξ and η subsequently. For a set of vertices v_1, v_2, \dots, v_k in G , define $G|v_1, \dots, v_k$ to be the subgraph of G induced on the set of vertices, each of which is smaller or equal to some v_i , $i = 1, 2, \dots, k$.

G is said to be a (c_1, c_2, \dots, c_l) -graph if every edge in G belongs to some α_i -path, or, in loose terms, all α_i 's "cover" the whole G . For a (c_1, c_2, \dots, c_l) -graph, the number of mergings is the number of vertices whose in-degree is at least 2. It is clear that to compute $\mathcal{M}(c_1, c_2, \dots, c_l)$ ($\mathcal{M}^*(c_1, c_2, \dots, c_l)$), it is enough to consider all the (c_1, c_2, \dots, c_l) -graphs with distinct (identical) sources. For a (c_1, c_2, \dots, c_l) -graph G , we say α_i is *reroutable* if there exists a different set of Menger's paths α'_i from S_i to R_i , and we say G is *reroutable* (or alternatively, there is a *rerouting* in G), if some α_i , $i = 1, 2, \dots, l$, is reroutable. Note that for a non-reroutable G , the choice of α_i 's is unique, so we often write $|G|_{\mathcal{M}}(\alpha_1, \alpha_2, \dots, \alpha_l)$ as $|G|_{\mathcal{M}}$ for notational simplicity.

Now, for a fixed i , reverse the directions of edges that only belong to α_i to obtain a new graph G' . For any two mergings λ, μ , if there exists a directed path in G' from the head (or tail) of λ to the head (or tail) of μ , we say the head (or tail) of λ *semi-reaches* the head (or

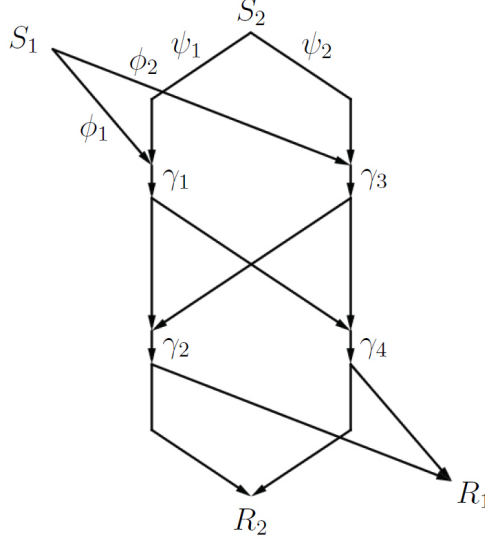


Figure 3: An example of a reroutable graph

tail) of μ against α_i , or alternatively, λ *semi-reaches* against α_i from head (or tail) to head (or tail). It is easy to check that G is reroutable if and only if there exists i and a merging λ such that λ semi-reaches itself against α_i from head to head, which is equivalent to the condition that there exists i and a merging η such that η semi-reaches itself against α_i from tail to tail.

Example 1.1. For the graph depicted in Figure 3, the source S_1 is connected to the sink R_1 by a group of Menger's paths

$$\begin{aligned} \phi = \{\phi_1, \phi_2\} = & \{S_1 \rightarrow h(\gamma_1) \rightarrow t(\gamma_1) \rightarrow h(\gamma_4) \rightarrow t(\gamma_4) \rightarrow R_1, \\ & S_1 \rightarrow h(\gamma_3) \rightarrow t(\gamma_3) \rightarrow h(\gamma_2) \rightarrow t(\gamma_2) \rightarrow R_1\} \end{aligned}$$

and the source S_2 is connected to the sink R_2 by a group of Menger's paths

$$\begin{aligned} \psi = \{\psi_1, \psi_2\} = & \{S_2 \rightarrow h(\gamma_1) \rightarrow t(\gamma_1) \rightarrow h(\gamma_2) \rightarrow t(\gamma_2) \rightarrow R_2, \\ & S_2 \rightarrow h(\gamma_3) \rightarrow t(\gamma_3) \rightarrow h(\gamma_4) \rightarrow t(\gamma_4) \rightarrow R_2\}. \end{aligned}$$

Then $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are mergings by ϕ -paths and ψ -paths. γ_1, γ_3 are smaller than γ_2 and γ_4 . $G|S_1, S_2)$ only consists of two isolated vertices S_1, S_2 ; $G|h(\gamma_1), h(\gamma_3))$ is the subgraph of G induced on the set of vertices $\{S_1, S_2, h(\gamma_1), h(\gamma_3)\}$; $G|t(\gamma_2), t(\gamma_4))$ is the subgraph of G induced on the set of vertices

$$\{S_1, S_2, h(\gamma_1), h(\gamma_3), t(\gamma_1), t(\gamma_3), h(\gamma_2), h(\gamma_4), t(\gamma_2), t(\gamma_4)\};$$

and $G|R_1, R_2)$ is just G itself.

The group of Menger's paths ϕ is reroutable, since there exists another group of Menger's paths

$$\begin{aligned} \phi' = \{\phi'_1, \phi'_2\} = & \{S_1 \rightarrow h(\gamma_1) \rightarrow t(\gamma_1) \rightarrow h(\gamma_2) \rightarrow t(\gamma_2) \rightarrow R_1, \\ & S_1 \rightarrow h(\gamma_3) \rightarrow t(\gamma_3) \rightarrow h(\gamma_4) \rightarrow t(\gamma_4) \rightarrow R_1\} \end{aligned}$$

from S_1 to R_1 . Similarly, ψ is also reroutable. It is easy to check, by definition, that γ_2 semi-reaches γ_4 against ψ from head to head, γ_1 semi-reaches γ_4 against ψ from tail to head, and γ_4 semi-reaches itself against ϕ (or alternatively ψ) from head to head. Hence, G is reroutable.

2 Related Sequences

2.1 Merging sequences

For any m, n , consider the following procedure to “draw” an (m, n) -graph: for “fixed” edge-disjoint paths $\psi_1, \psi_2, \dots, \psi_n$ from S_2 to R_2 , we extend edge-disjoint paths $\phi_1, \phi_2, \dots, \phi_m$ from S_1 to merge with ψ -paths until we reach R_1 . More specifically, the procedure of extending ϕ -paths is done step by step, and for each step, we choose to extend one of m ϕ -paths to merge with one of n ψ -paths. Thus for each step, we have mn “strokes” to choose from the following set

$$\{(\phi_1, \psi_1), (\phi_1, \psi_2), \dots, (\phi_m, \psi_{n-1}), (\phi_m, \psi_n)\},$$

here, by “drawing” the *path pair* (ϕ_i, ψ_j) , we mean further extending path ϕ_i to merge with path ψ_j , while ensuring the new merged subpath is larger than any existing merged subpaths on path ψ_j . Apparently, the procedure, and thus the graph, is uniquely determined by the sequence of strokes (see Example 2.1), which will be referred to as a *merging sequence* of this (m, n) -graph. It is also easy to see that any (m, n) -graph can be generated by some merging sequence.

Example 2.1. Consider the following two graphs in Figure 4 (here and hereafter, all the mergings in this paper are represented by solid dots instead). Listing the elements in the merging sequence, graph (a) can be described by $[(\phi_1, \psi_2), (\phi_2, \psi_1)]$, or alternatively $[(\phi_2, \psi_1), (\phi_1, \psi_2)]$. When the context is clear, we often omit ϕ, ψ in the merging sequence for notational simplicity. For example, graph (b) can be described by a merging sequence $[(1, 1), (2, 1), (2, 2), (3, 2)]$. Note that it cannot be described by $[(1, 1), (2, 1), (3, 2), (2, 2)]$, since $(3, 2)$ (or, more precisely, the merging corresponding to $(3, 2)$) is larger than $(2, 2)$ on ψ_2 .

2.2 AA-sequences

Consider a non-reroutable (m, n) -graph G with two sources S_1, S_2 , two distinct sinks R_1, R_2 , a set of Menger’s paths $\phi = \{\phi_1, \phi_2, \dots, \phi_m\}$ from S_1 to R_1 , and a set of Menger’s paths $\psi = \{\psi_1, \psi_2, \dots, \psi_n\}$ from S_2 to R_2 .

For the case when S_1 and S_2 are distinct, consider the following procedure on G . Starting from S_1 , go along path ϕ_i until we reach a merged subpath (or more precisely, the terminal vertex of a merged subpath), we then go against the associated ψ -path (corresponding to the merged subpath just visited) until we reach another merged subpath, we then go along the associated ϕ -path, ... Continue this procedure (of alternately going along ϕ -paths or going against ψ -paths until we reach a merged subpath) in the same manner as above, then the fact that G is non-reroutable and acyclic guarantees that eventually we will reach R_1 or

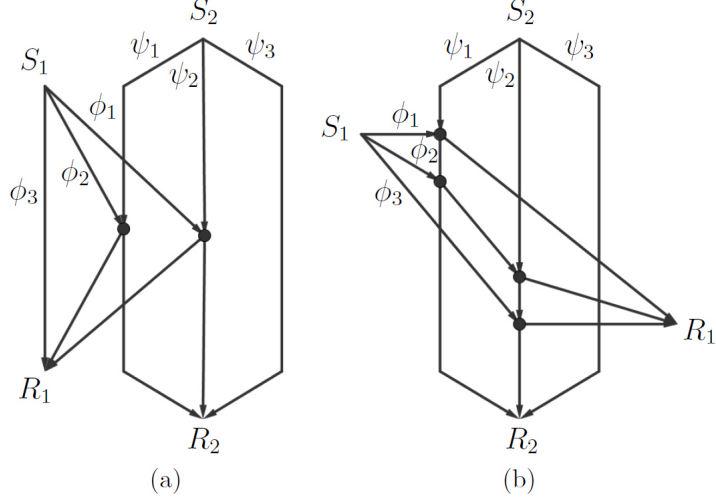


Figure 4: Two examples of merging sequences

S_2 . By sequentially listing all the terminal vertices of any merged subpaths visited, such a procedure produces a ϕ_i -AA-sequence. Apparently, there are m ϕ -AA-sequences.

Similarly, consider the following procedure on G . Starting from R_2 , go against path ψ_j until we reach a merged subpath, we then go along the associated ϕ -path (corresponding to the merged subpath just visited) until we reach another merged subpath, we then go against the associated ψ -path, ... Continue this procedure in the same manner, again, eventually, we are guaranteed to reach R_1 or S_2 . By sequentially listing all the terminal vertices of any merged subpaths visited, such a procedure produces a ψ_j -AA-sequence. Apparently, there are n ψ -AA-sequences.

The *length* of an AA-sequence π , denoted by $\text{Length}(\pi)$, is defined to be the number of terminal vertices of merged subpaths visited during the procedure. Since each such terminal vertex in an AA-sequence is associated with a path pair, equivalently, the length of an AA-sequence can be also defined as the number of the associated path pairs. For the purpose of computing $\mathcal{M}(m, n)$, we can assume that each Menger's path in G merges at least once, which implies that each AA-sequence is of positive length.

For the case when S_1 and S_2 are identical, by Proposition 3.6 in [5], we can restrict our attention to the case when $m = n$. For the purpose of computing $\mathcal{M}^*(n, n)$, by the proof of Proposition 3.6 in [5], we can assume that paths ϕ_i and ψ_i share a *starting subpath* (a maximal shared subpath by ϕ_i and ψ_i starting from the source) for $i = 1, 2, \dots, n$, and due to non-reroutability of G , ϕ_n and ψ_1 do not merge with any other path. Then, ψ -AA-sequences and their lengths can be similarly defined as in the case when S_1 and S_2 are distinct, except that we have to replace “merged subpath” by “merged subpath or starting subpath”. (Here, let us note that the procedure of defining ϕ -AA-sequences does NOT carry over.) It can be checked that the existence of m starting subpaths implies that any ψ -AA-sequence is of positive length and will always terminate at R_1 .

It turns out that the lengths of AA-sequences are related to the number of mergings in G .

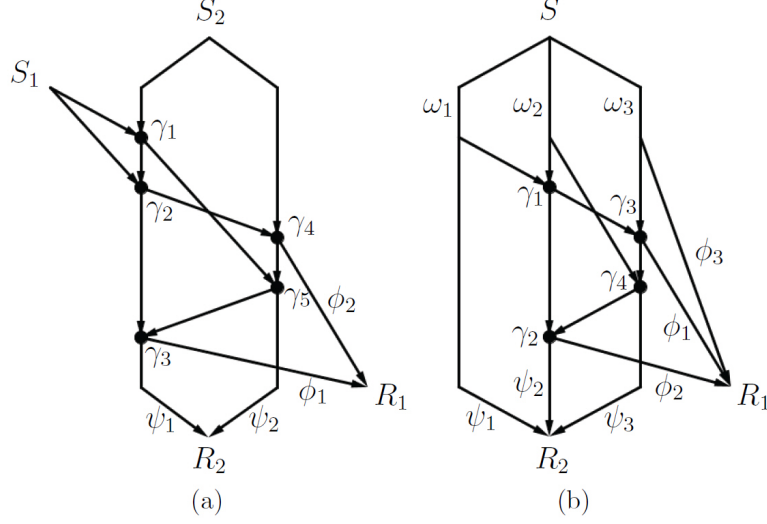


Figure 5: Two examples of AA-sequences

Lemma 2.2. *For a non-reroutable (m, n) -graph G with distinct sources,*

$$|G|_{\mathcal{M}} = \frac{1}{2} \sum_{\pi} \text{Length}(\pi);$$

for a non-reroutable (n, n) -graph G with identical sources and n starting subpaths,

$$|G|_{\mathcal{M}} = \frac{1}{2} \left(\sum_{\pi} \text{Length}(\pi) - n \right),$$

where the two summations above are over the all the possible AA-sequences.

Example 2.3. Consider the two graphs in Figure 5. Let “ \Rightarrow ” and “ \Leftarrow ” denote “go along” and “go against”, respectively. In graph (a), sequentially listing the terminal vertices of merged subpaths visited during the procedure, two ϕ -AA-sequences can be represented by $S_1 \Rightarrow h(\gamma_1) \Leftarrow S_2$ and $S_1 \Rightarrow h(\gamma_2) \Leftarrow t(\gamma_1) \Rightarrow h(\gamma_5) \Leftarrow t(\gamma_4) \Rightarrow R_1$. Similarly, two ψ -AA-sequences can be represented by $R_2 \Leftarrow t(\gamma_3) \Rightarrow R_1$ and $R_2 \Leftarrow t(\gamma_5) \Rightarrow h(\gamma_3) \Leftarrow t(\gamma_2) \Rightarrow h(\gamma_4) \Leftarrow S_2$. One also checks that the number of mergings is 5, which is half of $(1 + 4 + 1 + 4)$, the sum of lengths of all AA-sequences.

In graph (b), sequentially listing the terminal vertices of merged subpaths and starting subpaths visited during the procedure, three ψ -AA-sequences can be represented by $R_2 \Leftarrow t(\omega_1) \Rightarrow h(\gamma_1) \Leftarrow t(\omega_2) \Rightarrow h(\gamma_4) \Leftarrow t(\gamma_3) \Rightarrow R_1$, $R_2 \Leftarrow t(\gamma_2) \Rightarrow R_1$ and $R_2 \Leftarrow t(\gamma_4) \Rightarrow h(\gamma_2) \Leftarrow t(\gamma_1) \Rightarrow h(\gamma_3) \Leftarrow t(\omega_3) \Rightarrow R_1$. One also checks the number of mergings is 4, which is half of $(5 + 1 + 5 - 3)$.

Lemma 2.4. *The shortest ϕ -AA-sequence (ψ -AA-sequence) is of length at most 1.*

Proof. Suppose, by contradiction, that the shortest ϕ -AA-sequence is of length at least 2. Pick any ϕ -path, say ϕ_{i_0} . Assume that ϕ_{i_0} first merges with ψ_{j_0} at merged subpath λ_{i_0, j_0} . Since the ϕ_{i_0} -AA-sequence is of length at least 2, there exists a ϕ -path, say ϕ_{i_1} , such that

ϕ_{i_1} has a merged subpath, say μ_{i_1, j_0} , smaller than λ_{i_0, j_0} on ψ_{j_0} . Now assume that ϕ_{i_1} first merges with ψ_{j_1} at merged subpath λ_{i_1, j_1} , then similarly there exists a ϕ -path, say ϕ_{i_2} , such that ϕ_{i_2} has a merged subpath, say μ_{i_2, j_1} , smaller than λ_{i_1, j_1} on ψ_{j_1} . Continue this procedure in the similar manner to obtain $\psi_{j_2}, \lambda_{i_2, j_2}, \phi_{i_3}, \mu_{i_3, j_2}, \psi_{j_3}, \lambda_{i_3, j_3}, \phi_{i_4}, \mu_{i_4, j_3}, \dots$. Apparently, there exists $k < l$ such that $i_l = i_k$. One then checks that

$$\begin{aligned} & \phi_{i_k} [h(\lambda_{i_k, j_k}), h(\mu_{i_l, j_{l-1}})] \circ \psi_{j_{l-1}} [h(\mu_{i_l, j_{l-1}}), h(\lambda_{i_{l-1}, j_{l-1}})] \circ \phi_{i_{l-1}} [h(\lambda_{i_{l-1}, j_{l-1}}), h(\mu_{i_{l-1}, j_{l-2}})] \\ & \circ \psi_{j_{l-2}} [h(\mu_{i_{l-1}, j_{l-2}}), h(\lambda_{i_{l-2}, j_{l-2}})] \circ \dots \circ \phi_{i_{k+1}} [h(\lambda_{i_{k+1}, j_{k+1}}), h(\mu_{i_{k+1}, j_k})] \circ \psi_{j_k} [h(\mu_{i_{k+1}, j_k}), h(\lambda_{i_k, j_k})] \end{aligned}$$

constitutes a cycle, which contradicts the assumption that G is acyclic.

A parallel argument can be applied to the shortest ψ -AA-sequence. □

Lemma 2.5. *For a non-reroutable graph G , any path pair occurs at most once in any given AA-sequence.*

Proof. By contradiction, suppose that the same path pair occurs in an AA-sequence twice. As in the proof of Lemma 2.7 in [5], one can prove G is reroutable, which is a contradiction. □

Remark 2.6. It then immediately follows from Lemma 2.5 that in a non-reroutable (m, n) -graph with distinct sources,

- the longest ϕ -AA-sequence (ψ -AA-sequence) is of length at most mn ;
- any ϕ -path (ψ -path) merges at most mn times.

And in a non-reroutable (m, m) -graph with identical sources,

- the longest ψ -AA-sequence is of length at most m^2 ;
- any ϕ -path (ψ -path) merges at most m^2 times.

3 Exact Values

In this section, we give exact values of \mathcal{M} and \mathcal{M}^* for certain special parameters.

Theorem 3.1.

$$\mathcal{M}(2, n) = 3n - 1.$$

Proof. We first show that $\mathcal{M}(2, n) \geq 3n - 1$. Consider the following $(2, n)$ -graph specified by the following merging sequence (for a simple example, see Figure 6(a)): $\Omega = [\Omega_k : 1 \leq k \leq 3n - 1]$, where

$$\Omega_k = \begin{cases} ([i]_2, 1) & \text{if } k = 3i - 2 & \text{for } 1 \leq i \leq n, \\ ([i]_2, i + 1) & \text{if } k = 3i - 1 & \text{for } 1 \leq i \leq n - 1, \\ ([i + 1]_2, i + 1) & \text{if } k = 3i & \text{for } 1 \leq i \leq n - 1, \\ ([n + 1]_2, 1) & \text{if } k = 3n - 1. \end{cases}$$

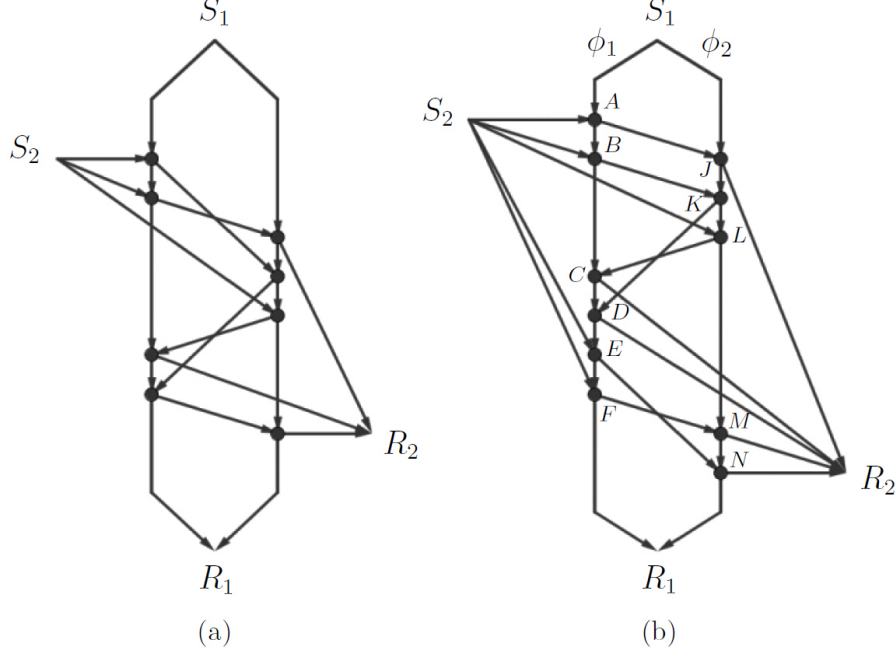


Figure 6: (a) A non-reroutable $(2, 3)$ -graph with 8 mergings (b) An example of a $(2, 5)$ -graph

where $[x]_2 = 1$ when x is odd, $[x]_2 = 2$ when x is even.

One checks that the above graph is non-reroutable with $3n - 1$ mergings, which implies that $\mathcal{M}(2, n) \geq 3n - 1$.

Next, we show that $\mathcal{M}(2, n) \leq 3n - 1$. Consider a non-reroutable $(2, n)$ -graph G with distinct sources S_1, S_2 , sinks R_1, R_2 , a set of Menger's paths $\phi = \{\phi_1, \phi_2\}$ from S_1 to R_1 , and a set of Menger's paths $\psi = \{\psi_1, \psi_2, \dots, \psi_n\}$ from S_2 to R_2 . Define

$$\Sigma = \{(\lambda, \mu) : \text{merging } \lambda \text{ is smaller than merging } \mu \text{ on some } \psi\text{-path} \\ \text{and there is no other merging between them on this path}\}.$$

Note that for any $(\lambda, \mu) \in \Sigma$, λ, μ must belong to different ϕ -paths. We say $(\lambda, \mu) \in \Sigma$ is of *type I*, if λ belongs to ϕ_1 , and $(\lambda, \mu) \in \Sigma$ is of *type II*, if λ belongs to ϕ_2 . For any two different elements $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \Sigma$. We say $(\lambda_1, \mu_1) \prec (\lambda_2, \mu_2)$ if either (they are of the same type and λ_1 is smaller than λ_2) or (they are of different types and λ_1 is smaller than μ_2). One then checks that the relationship defined by \prec is a strict total order.

Letting x denote the number of elements in Σ , we define

$$\Theta = (\Theta_1, \Theta_2, \dots, \Theta_x)$$

to be the sequence of the ordered (by \prec) elements in Σ . Now we consecutively partition Θ into z “medium-blocks” B_1, B_2, \dots, B_z , and further consecutively partition each B_i into y_i “mini-blocks” $B_{i,1}, B_{i,2}, \dots, B_{i,y_i}$ (see Example 3.3 for an example) such that

- for any i, j , the elements in $B_{i,j}$ are of the same type.
- for any i, j , $B_{i,j}$ is *linked* to $B_{i,j+1}$ in the following sense: let $(\lambda_{i,j}, \mu_{i,j})$ denote the element with the largest second component in $B_{i,j}$ and let $(\lambda_{i,j+1}, \mu_{i,j+1})$ denote the element with the smallest first component in $B_{i,j+1}$, then $\mu_{i,j} = \lambda_{i,j+1}$.

- for any i , B_{i,y_i} is not linked to $B_{i+1,1}$.

A mini-block is said to be a *singleton* if it has only one element. We then have the following lemma, whose proof is omitted.

Lemma 3.2. *Between any two “adjacent” singletons (meaning there is no singleton between these two singletons) in a medium-block, there must exist a mini-block containing at least three elements.*

Letting y denote the number of mini-blocks in Θ and x_i denote the number of elements in medium-block B_i for $1 \leq i \leq z$, we then have

$$\begin{aligned} x &= x_1 + x_2 + \cdots + x_z, \\ y &= y_1 + y_2 + \cdots + y_z. \end{aligned}$$

Suppose there are k singletons in Θ , then by Lemma 3.2, we can find $(k-1)$ mini-blocks, each of which has at least three elements. Hence, for $1 \leq i \leq z$,

$$x_i \geq 1 \cdot k + 3 \cdot (k-1) + 2 \cdot [y_i - k - (k-1)] = 2y_i - 1, \quad (1)$$

which implies

$$x = \sum_{i=1}^z x_i \geq \sum_{i=1}^z (2y_i - 1) = 2y - z. \quad (2)$$

For any two linked mini-blocks $B_{i,j}$ and $B_{i,j+1}$, let $(\lambda_{i,j}, \mu_{i,j})$ denote the element with the largest second component in $B_{i,j}$, and let $(\lambda_{i,j+1}, \mu_{i,j+1})$ denote the element with the smallest first component in $B_{i,j+1}$. By the definition (of two mini-blocks being linked), we have $\mu_{i,j} = \lambda_{i,j+1}$, which means $B_{i,j}$ and $B_{i,j+1}$ share a common merging. Together with the fact that each element in Σ is a pair of mergings, this further implies that the number of mergings in G is

$$|G|_{\mathcal{M}} = 2x - (y - z). \quad (3)$$

Notice that $\lambda_{i,j}, \lambda_{i,j+1}, \mu_{i,j+1}$ belong to the same ψ -path, and furthermore, there exists only one ϕ -path passing by both an element (more precisely, passing by both its mergings) in $B_{i,j}$ and an element in $B_{i,j+1}$. So, n , the number of ψ -paths in G can be computed as

$$n = x - (y - z). \quad (4)$$

It then follows from (2), (3), (4) and the fact $t \geq 1$ that

$$n = x - y + z \geq (2y - z) - y + z = y \quad (5)$$

and furthermore

$$|G|_{\mathcal{M}} = 2x - y + z = 2n + y - z \leq 2n + n - 1 = 3n - 1, \quad (6)$$

which establishes the theorem. \square

Example 3.3. Consider the graph in Figure 6(b) and assume the context is as in the proof of Theorem 3.1. Then we have,

$$\Sigma = \{(A, J), (B, K), (L, C), (K, D), (F, M), (E, N)\}.$$

Among all the elements in Σ , (A, J) , (B, K) , (F, M) and (E, N) are of type I, and (L, C) , (K, D) are of type II. It is easy to check that

$$\Theta = ((A, J), (B, K), (K, D), (L, C), (E, N), (F, M)),$$

which is partitioned into three mini-blocks $((A, J), (B, K))$, $((K, D), (L, C))$ and $((E, N), (F, M))$. The first mini-block is linked to the second one, but the second one is not linked to the third, so Θ is partitioned into two medium-blocks:

$$((A, J), (B, K), (K, D), (L, C)) \text{ and } ((E, N), (F, M)).$$

Remark 3.4. The result in Theorem 3.1 in fact has already been proved in [5] using a different approach. The proof in this paper, however, is more intrinsic in the sense that it reveals in greater depth the topological structure of non-reroutable $(2, n)$ -graphs achieving $3n - 1$ mergings, and further helps to determine the number of such graphs.

Assume a non-reroutable $(2, n)$ -graph G has $3n - 1$ mergings. One then checks that in the proof of Theorem 3.1, equalities hold for (6). It then follows that

- $t = 1$, namely, there is only one medium-block in Θ ;
- equalities hold necessarily for (5), (2) and eventually (1), which further implies that between two adjacent singletons, only one mini-block has three elements and any other mini-block has two elements.

Furthermore, one checks that

- for a mini-block with two elements $((\lambda_1, \mu_1), (\lambda_2, \mu_2))$, μ_2 is smaller than μ_1 ;
- for a mini-block with three elements $((\lambda_1, \mu_1), (\lambda_2, \mu_2), (\lambda_3, \mu_3))$, either $(\mu_2$ is smaller than μ_3 and μ_3 is smaller than $\mu_1)$ or $(\mu_3$ is smaller than μ_1 and μ_1 is smaller than $\mu_2)$.

Assume that G is “reduced” in the sense that, other than S_1, S_2, R_1, R_2 , each vertex in G is a terminal vertex of some merging. The properties above allow us to count how many reduced non-reroutable $(2, n)$ -graphs (up to graph isomorphism) can achieve $3n - 1$ mergings: suppose that there are k ($1 \leq k \leq \lfloor \frac{n+1}{2} \rfloor$) singletons in G , then necessarily, there are $(k - 1)$ three-element mini-blocks and $(n - 2k + 1)$ two-element mini-blocks in Θ . It can be checked that the number of ways for these n mini-blocks to form Θ for some $(2, n)$ -graph is $\binom{n}{2k-1} 2^{k-1}$. This implies that the number of $(2, n)$ -graph, whose Θ consists of k singletons, $(k - 1)$ three element mini-blocks and $(n - 2k + 1)$ two element mini-blocks, is $\binom{n}{2k-1} 2^{k-1}$. Through a computation summing over all feasible k , the number of reduced non-reroutable $(2, n)$ -graphs with $3n - 1$ mergings can be computed as

$$\sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n}{2k-1} 2^{k-1} = \frac{1}{2\sqrt{2}} [(1 + \sqrt{2})^n - (1 - \sqrt{2})^n] = P_n,$$

where P_n is the n -th Pell number [1].

Theorem 3.5.

$$\mathcal{M}^*(4, 4) = 9.$$

Proof. Consider a non-reroutable $(4, 4)$ -graph G with one source S , two sinks R_1, R_2 , a set of Menger's paths $\phi = \{\phi_1, \phi_2, \phi_3, \phi_4\}$ from S to R_1 and a set of Menger's paths $\psi = \{\psi_1, \psi_2, \psi_3, \psi_4\}$ from S to R_2 . As discussed in Section 2.2, we assume that ψ_i and ϕ_i share a starting subpath ω_i from S for $i = 1, 2, 3, 4$, and furthermore, we assume ϕ_4, ψ_1 do not merge with any other paths, directly “flowing” to the sinks.

Consider the four ψ -AA-sequences, which will be referred to as $\pi_1, \pi_2, \pi_3, \pi_4$ in the following. It is easy to check that each π_i , $i = 1, 2, 3, 4$, will be of odd length. Without loss of generality, assume that π_4 is the shortest such sequence, and thus by Lemma 2.4, π_4 is of length 1; let σ be the merging associated with π_4 . By Lemma 2.5, each π_i can only be associated with each path pair (ϕ_j, ψ_k) , $j = 1, 2, 3$ and $k = 2, 3, 4$ at most once. It then follows that excluding σ , each ψ_k , $k = 2, 3, 4$, can only merge with each ϕ_j , $j = 1, 2, 3$, at most once. One then further checks that each π_i , $i = 1, 2, 3$, can only be associated with (ϕ_j, ψ_k) , $j = 1, 2, 3$ and $k = 2, 3, 4$ for 7 times in total. By Lemma 2.2, we then derive

$$|G|_{\mathcal{M}} \leq (9 + 7 + 7 + 1 - 4)/2 = 10.$$

We next prove that $|G|_{\mathcal{M}}$ cannot be 10. Suppose, by contradiction, that $|G|_{\mathcal{M}}$ is 10. Then, necessarily, the longest ψ -AA-sequence, say π_1 , will be of length 9. It then follows that the two pairs, (ϕ_1, ψ_1) and (ϕ_4, ψ_4) must be associated with π_1 . It also follows that π_2, π_3 must be of length 7.

Now we prove that σ belongs to ϕ_1 and ψ_4 . It suffices to prove that each of $\psi_2, \psi_3, \phi_2, \phi_3$ cannot have four mergings. Suppose, by contradiction, there are four mergings in ψ_2 , say $\mu_1, \mu_2, \mu_3, \mu_4$, in the ascending order; here μ_4 is necessarily σ . Then, there are two mergings belonging to the same ϕ -path, say ϕ_k , $k \neq 2, 4$. Now we consider two cases:

If $\mu_1, \mu_4 \in \phi_k$, then $h(\mu_1), t(\mu_1), h(\mu_4), t(\mu_4)$ must belong to different ψ -AA-sequences. Suppose $h(\mu_1) \in \pi_{j_1}$, $t(\mu_1) \in \pi_{j_2}$, $h(\mu_4) \in \pi_{j_3}$ and $t(\mu_4) \in \pi_{j_4}$, where $\{j_1, j_2, j_3, j_4\} = \{1, 2, 3, 4\}$. Note that $t(\mu_i)$ and $h(\mu_{i+1})$ belong to the same ψ -AA-sequence for $i = 1, 2, 3$, $h(\mu_1)$ and $t(\mu_2)$ belong to the same ψ -AA-sequence. This implies that $t(\mu_2) \in \pi_{j_1}$, $h(\mu_2) \in \pi_{j_2}$ and $t(\mu_3) \in \pi_{j_3}$. It then follows that $t(\mu_2), h(\mu_3)$ cannot belong to π_{j_2} or π_{j_3} , so it must belong to π_{j_1} . On the other hand, either μ_2 or μ_3 must belong to ϕ_2 , the same ϕ -path to which $t(\mu_2)$ belongs. Then (ϕ_2, ψ_2) occurs at least twice in π_{j_1} , which violates the Lemma 2.5.

If $\mu_2, \mu_4 \in \phi_k$, then $h(\mu_2), t(\mu_2), h(\mu_4), t(\mu_4)$ must belong to different ψ -AA-sequences. Suppose $h(\mu_2) \in \pi_{j_1}$, $t(\mu_2) \in \pi_{j_2}$, $h(\mu_4) \in \pi_{j_3}$ and $t(\mu_4) \in \pi_{j_4}$, where $\{j_1, j_2, j_3, j_4\} = \{1, 2, 3, 4\}$. Note that $t(\mu_i)$ and $h(\mu_{i+1})$ belong to the same ψ -AA-sequence for $i = 1, 2, 3$, $h(\mu_1)$ and $t(\mu_2)$ belong to the same ψ -AA-sequence. This implies that $t(\mu_1) \in \pi_{j_1}$, $h(\mu_3) \in \pi_{j_2}$ and $t(\mu_3) \in \pi_{j_3}$. In this case μ_3 must belong to ϕ_2 , the same path to which $t(\mu_2)$ belongs. It then follows that $t(\mu_2)$ cannot belong to π_{j_2}, π_{j_3} , so it must belong to π_{j_1} . But then we have $h(\mu_1), t(\mu_1) \in \pi_{j_1}$, which violates the Lemma 2.5.

Combining the above two cases, we conclude that there cannot be four mergings on ψ_2 . With a parallel argument applied to ϕ_2, ϕ_3, ψ_3 , we conclude that there are four mergings on ψ_4 , say $\gamma_1, \gamma_2, \gamma_3, \gamma_4 = \sigma$, in the ascending order.

Next, we examine all the following cases to show that $|G|_{\mathcal{M}}$ cannot be 10.

Case 1: paths ϕ_1 and ψ_4 merge at γ_1 and γ_4 . For this case, we have the following two subcases.

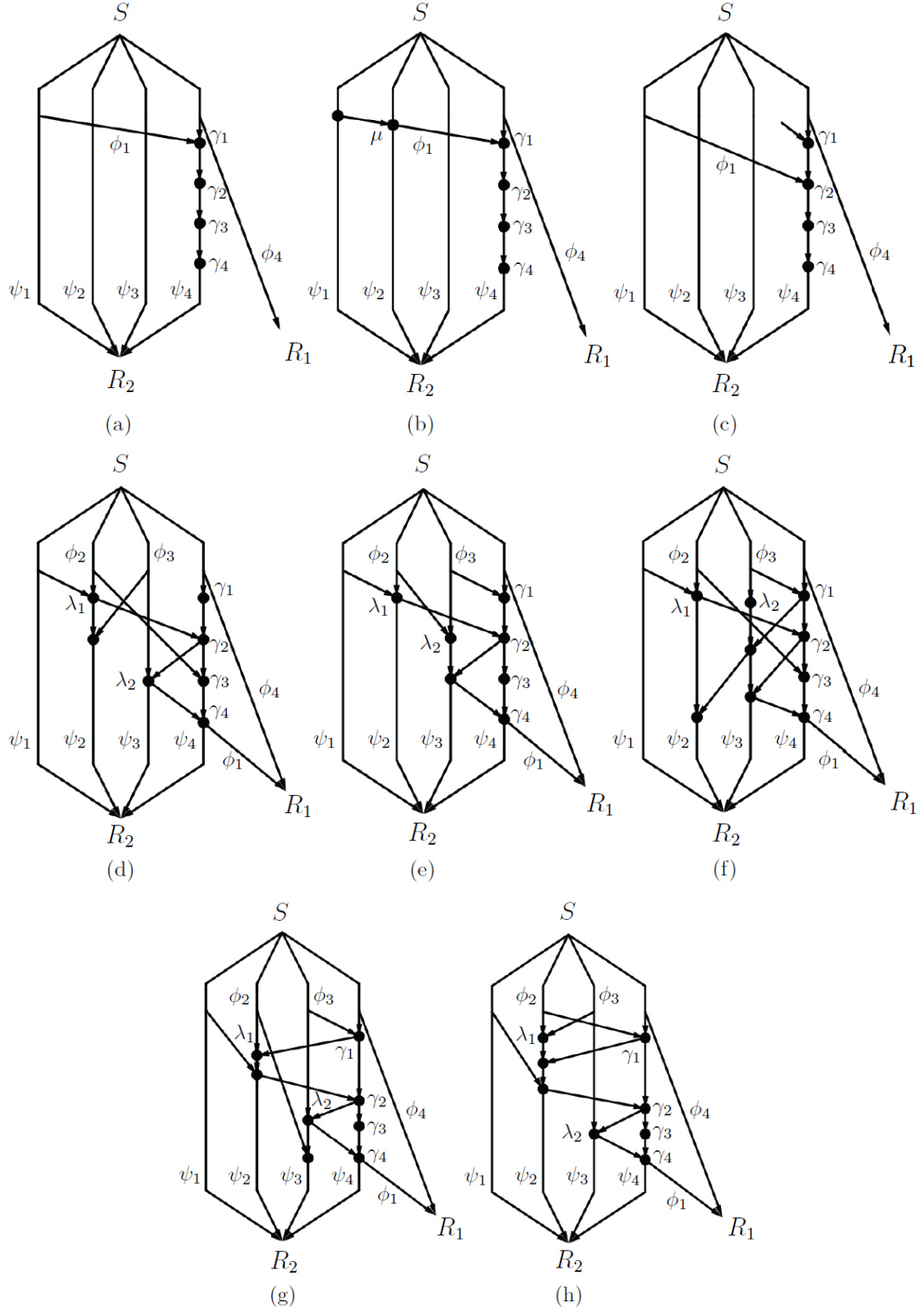


Figure 7: (a) Case 1.1 (b) Case 1.2 (c) Case 2.1 (d) Case 2.2.1 (e)(f) Case 2.2.2 (g)(h) Case 2.2.3

Case 1.1: ϕ_1 first merges with ψ_4 . For this case, it is easy to check that π_1 is of length 3, which contradicts the fact that it is of length 9 (see Figure 7(a)).

Case 1.2: ϕ_1 first merges with ψ_2 or ψ_3 ; without loss of generality, assume that ϕ_1 first merges with ψ_2 at the merging denoted by μ . Then necessarily, ϕ_1 immediately merges with ψ_4 at the merging γ_1 . Then, we have $h(\mu), t(\mu) \in \pi_1$, which violates Lemma 2.5 (see Figure 7(b)).

Case 2: paths ϕ_1 and ψ_4 merge at γ_2 and γ_4 . For this case, we consider the following subcases.

Case 2.1: ϕ_1 first merges with ψ_4 . Then, we have $h(\gamma_1), t(\gamma_1) \in \pi_1$, which violates Lemma 2.5 (see Figure 7(c)).

Case 2.2: ϕ_1 first merges with ψ_2 or ψ_3 ; without loss of generality, assume that ϕ_1 first merges with ψ_2 . Then necessarily, ϕ_1 will subsequently merges with ψ_4 , ψ_3 and ψ_4 . Let λ_1, λ_2 be the smallest mergings in ψ_2, ψ_3 , respectively. It is clear that at least one of λ_1 and λ_2 belongs to ϕ_1 , since otherwise λ_1, λ_2 would belong to ϕ_3, ϕ_2 , respectively, and thus λ_1 would semi-reach itself from head to head again ψ , which implies the existence of a rerouting, a contradiction.

Case 2.2.1: both the first mergings on ψ_2, ψ_3 belong to ϕ_1 . Then, γ_1 is the largest merging on either ϕ_2 or ϕ_3 , that is, from γ_1 , the associated path cannot go forward to merge anymore. ϕ_3 can only first merges with ψ_2 and ϕ_2 can only first merges with ψ_4 at γ_3 , which implies the existence of a rerouting (γ_3 semi-reaches itself against ϕ from head to head). See Figure 7(d) for an example.

Case 2.2.2: the first merging λ_1 on ψ_2 belongs to ϕ_1 , and the first merging on ψ_3 belongs to ϕ_2 . If ϕ_2 first merges with ψ_3 , then ϕ_3 can only first merges ψ_4 at γ_1 , one check that π_1 is of length 8, a contradiction (see Figure 7(e)); if ϕ_2 first merges with ψ_4 at γ_3 (if ϕ_2 first merges with ψ_4 at γ_1 , then π_1 is of length 6, a contradiction), then ϕ_3 can only first merge with ψ_4 at γ_1 , and then merges with ψ_3, ψ_2 , which implies the existence of a rerouting (γ_3 semi-reaches itself against ϕ from head to head). See Figure 7(f) for an example.

Case 2.2.3: the first merging λ_2 on ψ_3 belongs to ϕ_1 , and the first merging λ_1 on ψ_2 belongs to ϕ_3 . If ϕ_3 first merges with ψ_4 , then necessarily the merging is γ_1 , and ϕ_3 further merges with ψ_2 at λ_1 . In this case ϕ_2 cannot go forward to merge anymore, which contradicts the fact that ϕ_2 merges with ψ -paths just three times (see Figure 7(g)); if ϕ_3 first merges ψ_2 at λ_1 , then ϕ_2 can only first merges with ψ_4 at γ_1 , and then merge with ψ_2 . In this case, ϕ_2 cannot go forward to merge anymore, which also contradicts the fact that ϕ_2 merges with ψ -paths exactly three times (see Figure 7(h)).

All the above cases combined imply that $|G|_{\mathcal{M}}$ is at most 9. On the other hand, one can find a non-reroutable $(4, 4)$ -graph with one source, two sinks and 9 mergings as in Figure 14, which implies $|G|_{\mathcal{M}} \geq 9$ (see a more general result in Theorem 4.2). We then have established the theorem. \square

Theorem 3.6.

$$\mathcal{M}(3, 3) = 13.$$

Proof. Consider a non-reroutable $(3, 3)$ -graph G with two source S_1, S_2 and two sinks R_1, R_2 . Let $\phi = \{\phi_1, \phi_2, \phi_3\}$, $\psi = \{\psi_1, \psi_2, \psi_3\}$ denote the set of Menger's paths from S_1, S_2 to R_1, R_2 , respectively.

As discussed in Section 2.2, we assume each AA-sequences is of positive length. Then, by Lemma 2.4, the shortest AA-sequence is of length 1. It can also be checked that the longest

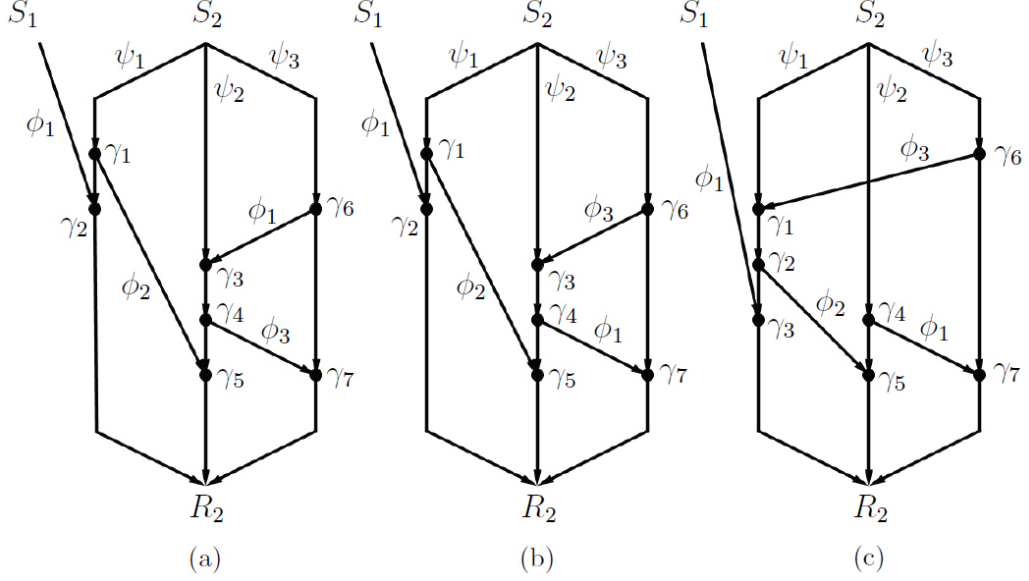


Figure 8: Three possible cases for the ϕ_1 -AA-sequence

AA-sequence in G is of length at most 7. So, by Lemma 2.2, we have

$$\mathcal{M}(3, 3) \leq (7 + 7 + 1 + 7 + 7 + 1)/2 = 15.$$

It follows from Theorem 4.6 (this is proven later in Section 4) that $\mathcal{M}(3, 3) \geq 13$. We next show $\mathcal{M}(3, 3)$ cannot be 15 or 14. Note that any non-reroutable $(3, 3)$ -graph having 15 mergings implies that

$$\text{its } (\phi\text{-AA-sequences}; \psi\text{-AA-sequences}) \text{ are of length } (7, 7, 1; 7, 7, 1), \text{ respectively;} \quad (7)$$

and 14 mergings implies that

$$\begin{aligned} &\text{its } (\phi\text{-AA-sequences}; \psi\text{-AA-sequences}) \text{ are of length } (7, 6, 1; 7, 6, 1), (7, 7, 1; 7, 5, 1), \\ &(7, 7, 1; 6, 6, 1), (7, 5, 1; 7, 7, 1), (6, 6, 1; 7, 7, 1), \text{ respectively.} \end{aligned} \quad (8)$$

The idea of the proof is that we first preprocess to eliminate many cases by checking if (7) and (8) are satisfied, then we can exhaustively investigate all the remaining cases to prove $\mathcal{M}(3, 3)$ cannot be equal to 14 or 15.

Suppose, by contradiction, that G has 14 or 15 mergings. Then, as before, at least one of AA-sequences of G is of length 7. Without loss of generality, we assume the ϕ_1 -AA-sequences is of length 7. One then checks that, up to obvious symmetry, as depicted in Figure 8, we only have three possible cases for the ϕ_1 -AA-sequence: for Case 1, the ϕ_1 -AA-sequence is $S_1 \Rightarrow h(\gamma_2) \Leftarrow t(\gamma_1) \Rightarrow h(\gamma_5) \Leftarrow t(\gamma_4) \Rightarrow h(\gamma_7) \Leftarrow t(\gamma_6) \Rightarrow h(\gamma_3) \Leftarrow S_2$; for Case 2, the ϕ_1 -AA-sequence is $S_1 \Rightarrow h(\gamma_2) \Leftarrow t(\gamma_1) \Rightarrow h(\gamma_5) \Leftarrow t(\gamma_4) \Rightarrow h(\gamma_7) \Leftarrow t(\gamma_6) \Rightarrow h(\gamma_3) \Leftarrow S_2$; for Case 3, the ϕ_1 -AA-sequence is $S_1 \Rightarrow h(\gamma_3) \Leftarrow t(\gamma_2) \Rightarrow h(\gamma_5) \Leftarrow t(\gamma_4) \Rightarrow h(\gamma_7) \Leftarrow t(\gamma_6) \Rightarrow h(\gamma_1) \Leftarrow S_2$.

Note that the graphs in Figure 8 only show the segments of paths ϕ_1, ϕ_2, ϕ_3 associated with the ϕ_1 -AA-sequence. Next, for each of the above-mentioned cases, we will extend these

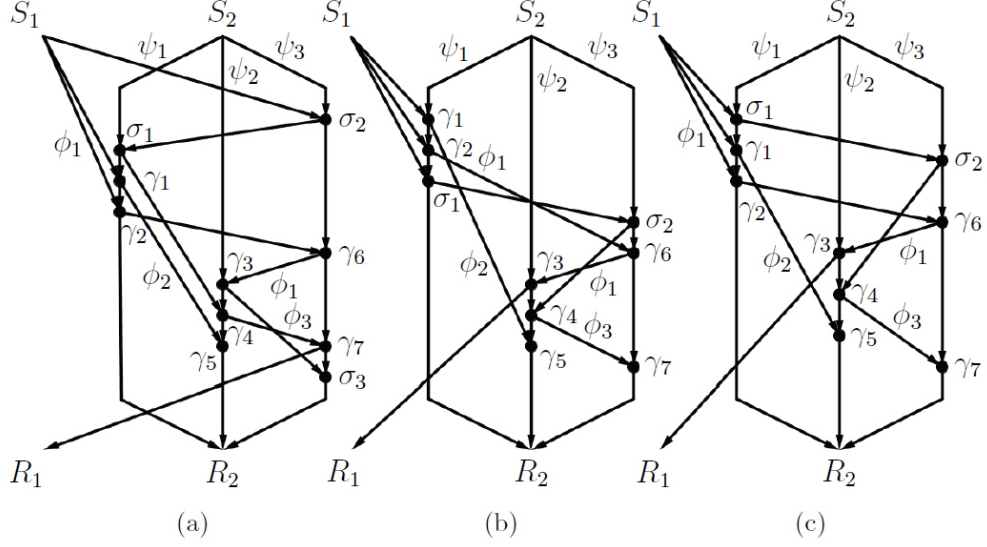


Figure 9: Case 1, Subcase (2, 0)

segments either backward or forward in all possible ways, and we shall show that no matter how we extend, the number of mergings in G will not exceed 13.

Case 1: as shown in Figure 8(a).

For this case, one checks that after ϕ_1 first merge with ψ_1 at γ_2 , it must immediately merge with ψ_3 at γ_6 ; one also checks that for paths ϕ_2, ϕ_3 , each of them can only go backward to merge at most twice. In the following, by Subcase (l_1, l_2) , we mean the case when path ϕ_3 goes backward to merge l_1 times and path ϕ_2 goes backward to merge l_2 times. It suffices to check the following nine subcases: $(0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$, $(1, 1)$, $(1, 2)$, $(2, 0)$, $(2, 1)$, $(2, 2)$.

The checking procedure is rather mechanical and tedious, so we only go through Subcase $(2, 0)$, as shown in Figure 9, for illustrative purposes. For this case, we have three choices for path ϕ_3 .

For Choice 1 as shown in Figure 9(a), the ϕ_3 -AA-sequence is of length 1, so path ϕ_1 must go forward to merge further to make sure the ϕ_2 -AA-sequence is of length more than 6. Therefore, from γ_7 , path ϕ_3 cannot go forward to merge any more and it must go to R_2 directly. Then one exhaustively checks that from γ_9 and γ_5 , paths ϕ_1 and ϕ_2 cannot go forward to merge more than four times in total.

For Choice 2 as shown in Figure 9(b), the ϕ_2 -AA-sequence is of length 1, and path ϕ_1 cannot go forward to merge anymore. One exhaustively checks that from γ_5 and γ_7 , paths ϕ_2 and ϕ_3 cannot go forward to merge five times in total.

For Choice 3 as shown in Figure 9(c), the ϕ_3 -AA-sequence is of length 1, and the ϕ_2 -AA-sequence is of length 3. So, (7) or (8) is not satisfied.

Case 2: as shown in Figure 8(b).

For this case, one checks that each of paths ϕ_2 and ϕ_3 cannot go backward to merge more than three times. One also checks that path ϕ_1 , after merging with ψ_1 at γ_2 , will immediately merge with ψ_2 at γ_4 . Since otherwise, one verifies that the total number of mergings is strictly less than 14: path ϕ_3 can go backward to merge for at most twice and

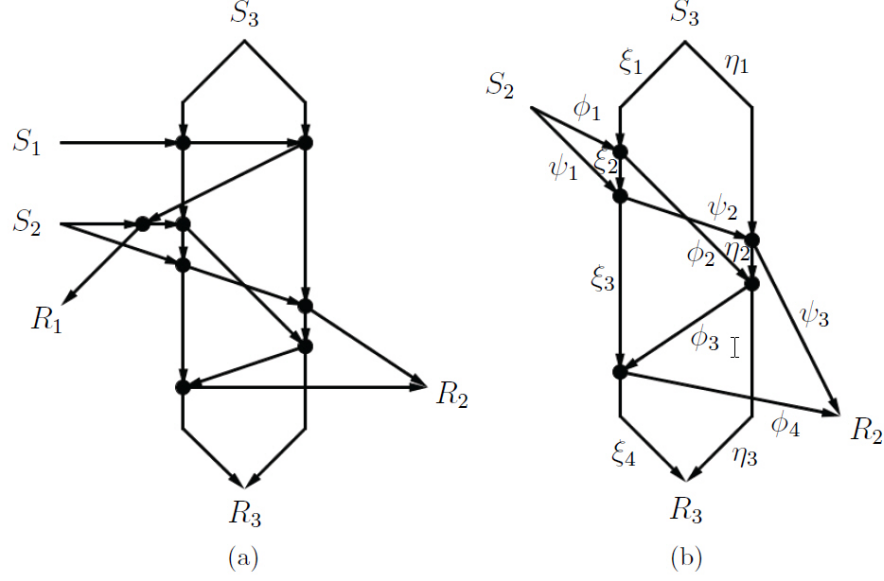


Figure 10: (a) A non-reroutable $(1, 2, 2)$ -graph with 8 mergings (b) The edge-labeled non-reroutable $(2, 2)$ -graph

path ϕ_2 cannot go backward to merge; furthermore, path ϕ_3 cannot go forward to merge anymore from γ_3 and paths ϕ_1 and ϕ_2 cannot go forward to merge four times in total. It suffices to check the following subcases: $(0, 0)$, $(1, 0)$, $(1, 1)$, $(2, 0)$, $(2, 1)$, $(2, 2)$, $(1, 3)$, $(0, 1)$, $(0, 2)$, $(1, 2)$.

Case 3: as shown in Figure 8(c).

For this case, after path ϕ_1 merges with ψ_1 at γ_3 , it has to immediately merge with ψ_2 at γ_4 . Similarly as before, it suffices to check the following subcases: $(0, 0)$, $(1, 0)$, $(1, 1)$, $(2, 1)$, $(2, 2)$, $(0, 1)$, $(0, 2)$, $(1, 2)$.

□

Theorem 3.7.

$$\mathcal{M}(1, 2, n) = \begin{cases} 4n & \text{if } n = 2, 3, \\ 4n + 1 & \text{if } n = 1 \text{ or } n \geq 4. \end{cases}$$

Proof. It follows from [5] that

$$\mathcal{M}(1, 2, n) \leq \mathcal{M}(1, 2) + \mathcal{M}(1, n) + \mathcal{M}(2, n) = 2 + n + (3n - 1) = 4n + 1. \quad (9)$$

To prove the theorem, we will consider the following four cases:

Case 1: $n = 1$. It immediately follows from Theorem 3.11 that

$$\mathcal{M}(1, 2, 1) = \mathcal{M}(1, 1, 2) = 5.$$

Case 2: $n = 2$. It can be checked that the $(1, 2, 2)$ -graph in Figure 10(a) is non-reroutable, which implies that $\mathcal{M}(1, 2, 2) \geq 8$. Since, by (9), $\mathcal{M}(1, 2, 2) \leq 9$, it suffices to prove that $\mathcal{M}(1, 2, 2)$ is not 9.

Suppose, by contradiction, that a non-reroutable $(1, 2, 2)$ -graph G has 9 mergings. Assume G has distinct sources S_1, S_2, S_3 , sinks R_1, R_2, R_3 , path β from S_1 to R_1 , a set of two Menger's paths $\{\phi, \psi\}$ from S_2 to R_2 and a set of two Menger's paths $\{\xi, \eta\}$ from S_3 to R_3 .

Since G is non-reroutable, path β merges with each of paths ϕ, ψ, ξ, η at most once (otherwise path β is reroutable through the path with which β merges twice). This, together with the assumption that $|G|_{\mathcal{M}} = 9$ and the fact that $\mathcal{M}(2, 2) = 5$, implies that $|G'|_{\mathcal{M}} = 5$, where G' denotes the subgraph of G induced on ϕ, ψ, ξ, η , and β must merge with each of ϕ, ψ, ξ, η exactly once. Here, by Remark 3.4, G' has only two “reduced” instances: the graph in Figure 10(b) and its “reversed” version obtained by reversing all its edges; so, we can assume G' takes the form as in Figure 10(b). Moreover, since we count mergings without multiplicity, we can further assume that every merging in G is by exactly two Menger’s paths.

Now, we exhaustively examine all ways in which β can merge with G' without generating any reroutings or cycles. The following rule can be used to eliminate many cases: For any two paths β', β'' in G' , if β' is smaller than β'' , then β cannot merge with them both (since, otherwise, β is reroutable through β' and β'').

With the subpaths of ϕ, ψ, ξ, η labeled as in Figure 10(b), we obtain the following sets of subpaths, each of which consists of (unordered) subpaths, where β can merge with ϕ, ψ, ξ, η without violating the above-mentioned rule: $\{\phi_1, \psi_1, \xi_1, \eta_1\}$, $\{\phi_2, \psi_1, \xi_2, \eta_1\}$, $\{\phi_2, \psi_2, \xi_3, \eta_1\}$, $\{\phi_2, \psi_3, \xi_3, \eta_2\}$, $\{\phi_3, \psi_3, \xi_3, \eta_3\}$, $\{\phi_4, \psi_3, \xi_4, \eta_3\}$. In the following, we examine each of the above possibilities, and conclude that there is no way one can add path β without generating reroutings or cycles, which further implies that $\mathcal{M}(1, 2, 2) = 8$.

Below, expression like “ $\eta_1 \mapsto \phi_1 \mapsto \psi_1 : \{\xi, \eta\}$ ” means “if after β merges with η_1 , it further immediately with ϕ_1 , and further immediately with ψ_1 , then the group of Menger’s paths $\{\xi, \eta\}$ are reroutable”.

1. $\phi_1, \psi_1, \xi_1, \eta_1$.

$\phi_1 \mapsto \xi_1 : \{\phi, \psi\}$, $\xi_1 \mapsto \phi_1 : \{\xi, \eta\}$, $\psi_1 \mapsto \eta_1 : \{\phi, \psi\}$, $\eta_1 \mapsto \psi_1 : \{\xi, \eta\}$,
 $\phi_1 \mapsto \eta_1 : \{\phi, \psi\}$, $\xi_1 \mapsto \psi_1 : \{\xi, \eta\}$, $\psi_1 \mapsto \xi_1 \mapsto \eta_1 : \{\phi, \psi\}$, $\eta_1 \mapsto \phi_1 \mapsto \psi_1 : \{\xi, \eta\}$.
It is easy to check we cannot find path β without some of the above subpaths.

2. $\phi_2, \psi_1, \xi_2, \eta_1$.

For ξ_2 and ϕ_2 , $\xi_2 \mapsto \phi_2 : \{\phi, \psi\}$, $\phi_2 \mapsto \xi_2 : \{\xi, \eta\}$.
For ξ_2 and ψ_1 , $\xi_2 \mapsto \psi_1 : \{\xi, \eta\}$, $\psi_1 \mapsto \xi_2 : \{\phi, \psi\}$.
For ξ_2 and η_1 , $\xi_2 \mapsto \eta_1 : \{\phi, \psi\}$, $\eta_1 \mapsto \xi_2 : \{\xi, \eta\}$.
Hence, path β cannot merge with ξ_2 , if it merges the other three edges.

3. $\phi_2, \psi_2, \xi_3, \eta_1$.

For η_1 and ϕ_2 , $\eta_1 \mapsto \phi_2 : \{\xi, \eta\}$, $\phi_2 \mapsto \eta_1 : \{\phi, \psi\}$.
For η_1 and ψ_2 , $\eta_1 \mapsto \psi_2 : \{\xi, \eta\}$, $\psi_2 \mapsto \eta_1 : \{\phi, \psi\}$.
For η_1 and ξ_3 , $\eta_1 \mapsto \xi_3 : \{\xi, \eta\}$, $\xi_3 \mapsto \eta_1 : \{\phi, \psi\}$.
Hence, path β cannot merge with η_1 , if it merges the other three edges.

4. $\phi_2, \psi_3, \xi_3, \eta_2$.

For ψ_3 and ϕ_2 , $\psi_3 \mapsto \phi_2 : \{\xi, \eta\}$, $\phi_2 \mapsto \psi_3 : \{\phi, \psi\}$.
For ψ_3 and ξ_3 , $\psi_3 \mapsto \xi_3 : \{\xi, \eta\}$, $\xi_3 \mapsto \psi_3 : \{\phi, \psi\}$.
For ψ_3 and η_2 , $\psi_3 \mapsto \eta_2 : \{\xi, \eta\}$, $\eta_2 \mapsto \psi_3 : \{\phi, \psi\}$.
Hence, path β cannot merge with ψ_3 , if it merges the other three edges.

5. $\phi_3, \psi_3, \xi_3, \eta_3$.

For ϕ_3 and ψ_3 , $\phi_3 \mapsto \psi_3 : \{\phi, \psi\}$, $\psi_3 \mapsto \phi_3 : \{\xi, \eta\}$.

For ϕ_3 and ξ_3 , $\phi_3 \mapsto \xi_3 : \{\phi, \psi\}$, $\xi_3 \mapsto \phi_3 : \{\xi, \eta\}$.

For ϕ_3 and η_3 , $\phi_3 \mapsto \eta_3 : \{\xi, \eta\}$, $\eta_3 \mapsto \phi_3 : \{\phi, \psi\}$.

Hence, path β cannot merge with ϕ_3 , if it merges the other three edges.

6. $\phi_4, \psi_3, \xi_4, \eta_3$.

$\phi_4 \mapsto \xi_4 : \{\xi, \eta\}$, $\xi_4 \mapsto \phi_4 : \{\phi, \psi\}$, $\psi_3 \mapsto \eta_3 : \{\xi, \eta\}$, $\eta_3 \mapsto \psi_3 : \{\phi, \psi\}$,

$\psi_3 \mapsto \xi_4 : \{\xi, \eta\}$, $\eta_3 \mapsto \phi_4 : \{\phi, \psi\}$, $\psi_3 \mapsto \phi_4 \mapsto \eta_3 : \{\xi, \eta\}$, $\eta_3 \mapsto \xi_4 \mapsto \psi_3 : \{\phi, \psi\}$.

It is easy to check we cannot find path β without some of the above subpaths.

Case 3: $n = 3$. It can be checked that the $(1, 2, 3)$ -graph as in Figure 11(a) is non-reroutable, which implies that $\mathcal{M}(1, 2, 3) \geq 12$. Since, by (9), $\mathcal{M}(1, 2, 3) \leq 13$, it suffices to prove that $\mathcal{M}(1, 2, 2)$ is not 13.

Suppose, by contradiction, that a non-reroutable $(1, 2, 3)$ -graph G has 13 mergings. Assume G has distinct sources S_1, S_2, S_3 , sinks R_1, R_2, R_3 , path β from S_1 to R_1 , a set of two Menger's paths $\{\phi, \psi\}$ from S_2 to R_2 and a set of three Menger's paths $\{\xi, \eta, \delta\}$ from S_3 to R_3 .

Since G is non-reroutable, path β merges each of paths $\phi, \psi, \xi, \eta, \delta$ at most once (otherwise path β is reroutable through the path with which p merges twice). This, together with the fact that $|G|_{\mathcal{M}} = 13$ and the fact that $\mathcal{M}(2, 3) = 8$, implies that β must merge with each of $\phi, \psi, \xi, \eta, \delta$ exactly once and the number of mergings among $\{\phi, \psi\}$ and $\{\xi, \eta, \delta\}$ is 8.

Similar to the proof for the case $n = 2$, we consider the subgraph G' of G induced on paths $\phi, \psi, \xi, \eta, \delta$. One then checks that any $(2, 3)$ -graph must have, up to relabeling, one of five merging sequences. We then exhaustively investigate how β can be "added" to G' to form G without generating any reroutings or cycles. Through a similar discussion, we conclude that there is no way we can add such path β to generate a non-reroutable $(1, 2, 3)$ -graph with 13 mergings. As a result, $\mathcal{M}(1, 2, 3) = 12$.

Case 4: $n \geq 4$. By (9), we only need to construct a non-reroutable $(1, 2, n)$ -graph with $4n+1$ mergings. First, we consider a non-reroutable $(2, n)$ -graph with distinct sources S_2, S_3 , sinks R_2, R_3 and the following merging sequence:

$$\Omega_1 = (2, 1), \Omega_2 = (1, 1), \Omega_3 = (1, 2), \Omega_4 = (2, 2), \Omega_5 = (1, 3), \Omega_6 = (2, 3), \Omega_7 = (2, 1);$$

for $8 \leq k \leq 3n - 1$,

$$\Omega_k = \begin{cases} ([i]_2, 1) & \text{if } k = 3i - 1 \quad \text{for } 3 \leq i \leq n, \\ ([i]_2, i + 1) & \text{if } k = 3i \quad \text{for } 3 \leq i \leq n - 1, \\ ([i + 1]_2, i + 1) & \text{if } k = 3i + 1 \quad \text{for } 3 \leq i \leq n - 1, \end{cases}$$

where $[x]_2 = 1$ when x is odd, $[x]_2 = 2$ when x is even. One can check that this $(2, n)$ -graph is non-reroutable.

Assume that the two Menger's paths from S_2 to R_2 start with the subpaths ξ_1, ξ_2 , respectively; and the n Menger's paths from S_3 to R_3 start with the subpaths $\eta_1, \eta_2, \dots, \eta_n$, respectively; and there are no mergings on $\xi_1, \xi_2, \eta_1, \eta_2, \dots, \eta_n$. Next, we add a path β to construct a non-reroutable $(1, 2, n)$ -graph such that path β , starting from S_1 , successively merges with $\eta_1, \eta_2, \dots, \eta_n, \xi_1, \xi_2$ (these mergings are labeled as $\mu_1, \mu_2, \dots, \mu_n, \lambda_1, \lambda_2$ in Figure 11(b)),

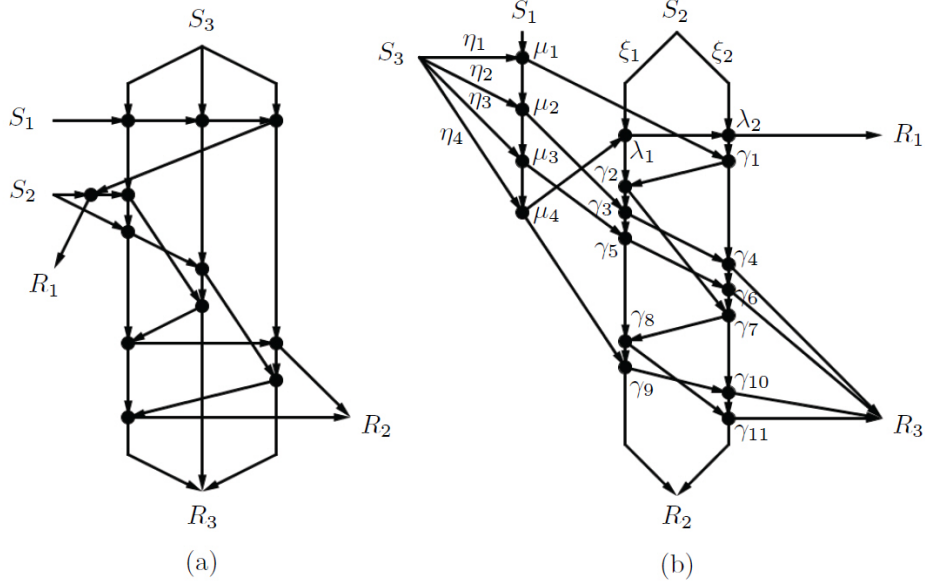


Figure 11: (a) A non-reroutable $(1,2,3)$ -graph with 12 mergings (b) A non-reroutable $(1,2,4)$ -graph with 17 mergings

and eventually reaches R_1 . It can be checked that this newly constructed $(1,2,n)$ -graph is non-reroutable. \square

Remark 3.8. Through exhaustive searching, we are able to compute exact values for \mathcal{M} and \mathcal{M}^* with some small parameters: $\mathcal{M}(3,4) = 18$, $\mathcal{M}(3,5) = 23$, $\mathcal{M}(3,6) = 28$, $\mathcal{M}(4,4) = 27$, $\mathcal{M}^*(5,5) = 16$, $\mathcal{M}^*(6,6) = 27$, $\mathcal{M}(2,2,2) = 11$, $\mathcal{M}(1,3,3) = 17$, $\mathcal{M}(2,2,3) = 18$, $\mathcal{M}^*(2,3,3) = 5$, $\mathcal{M}^*(2,4,4) = 10$, $\mathcal{M}^*(2,5,5) = 17$, $\mathcal{M}^*(3,3,3) = 8$, $\mathcal{M}^*(3,4,4) = 13$, $\mathcal{M}^*(4,4,4) = 18$. Computations show that for $m \leq n \leq n'$ and $(m,n) \leq (3,4)$ or $(2,5)$,

$$\mathcal{M}^*(m,n,n') = \mathcal{M}^*(m,n,n).$$

Theorem 3.9.

$$\mathcal{M}(\underbrace{1,1,\dots,1}_k) = \left\lfloor \frac{k^2}{4} \right\rfloor.$$

Proof. For the “ \geq ” direction, by Proposition 2.12 of [5], we deduce that

$$\mathcal{M}(\underbrace{1,1,\dots,1}_k) \geq \sum_{i \leq \lfloor k/2 \rfloor, j \geq \lfloor k/2 \rfloor + 1} \mathcal{M}(1,1) = \left\lfloor \frac{k^2}{4} \right\rfloor.$$

To prove the “ \leq ” direction, consider a non-reroutable $(1,1,\dots,1)$ -graph G with distinct sources and sets of Menger’s paths $\{\beta_1\}, \{\beta_2\}, \dots, \{\beta_k\}$. It is easy to check that due to non-reroutability of G , any two β -paths can merge with each other at most once. Without loss of generality, assume that β_k merges j times with $\beta_1, \beta_2, \dots, \beta_j$, $1 \leq j \leq k-1$; and any other path β_i , $i \neq k$, merges at most j times. Again, due to non-reroutability of G , there are

no non- β_k -involved mergings among paths $\beta_1, \beta_2, \dots, \beta_j$, where we say a merging at edge e is β_k -involved if e belongs to β_k . It then follows that any non- β_k -involved merging in G must be associated with one of paths from $\beta_{j+1}, \dots, \beta_k$, each of which merges at most j times. We then conclude that

$$|G|_{\mathcal{M}} \leq j + (k - j - 1)j = (k - j)j \leq \left\lfloor \frac{k^2}{4} \right\rfloor.$$

□

Remark 3.10. For a non-reroutable $(\underbrace{1, 1, \dots, 1}_k)$ -graph G , in order to prove

$$|G|_{\mathcal{M}} \leq \left\lfloor \frac{k^2}{4} \right\rfloor,$$

we only need the following two conditions:

1. any two β_{i_1}, β_{i_2} can merge at most once;
2. there are at most two mergings in any subgraph of G induced on any three $\beta_{i_1}, \beta_{i_2}, \beta_{i_3}$.

So, in some sense, Theorem 3.9 is a “dual” version of the classical Turan’s theorem [11], which states that the number of edges in a graph is less than $\lfloor \frac{k^2}{4} \rfloor$ if

1. the graph is simple, i.e., there is at most one edge between any two vertices;
2. the graph does not have triangles, i.e., there are at most two edges among any three vertices.

Theorem 3.11.

$$\mathcal{M}(\underbrace{1, \dots, 1}_k, 2) = \begin{cases} 3k - 1 & \text{if } k \leq 6, \\ \lfloor \frac{k^2}{4} \rfloor + k + 2 & \text{if } k > 6. \end{cases}$$

Proof. The upper bound direction: Consider any non-reroutable $(\underbrace{1, 1, \dots, 1}_k, 2)$ -graph G with

distinct sources $S_1, S_2, \dots, S_k, \widehat{S}$, sinks $R_1, R_2, \dots, R_k, \widehat{R}$, a Menger’s path β_i from S_i to R_i for $1 \leq i \leq k$, two Menger’s paths ψ_1, ψ_2 from \widehat{S} to \widehat{R} . Let $B_1(B_2)$ denote the set of β -paths, each of which first merges with $\psi_1(\psi_2)$ and then with $\psi_2(\psi_1)$. Let $A_1(A_2)$ denote the set of β -paths, each of which only merges with $\psi_1(\psi_2)$, and let C denote the set of β -paths, each of which does not merge with ψ_1 or ψ_2 . And we write

$$A = A_1 \cup A_2, \quad B = B_1 \cup B_2.$$

Consider any path β_k in B . Assume that β_k merges with ψ_1 at merged subpath $\gamma_{k,1}$ and with ψ_2 at merged subpath $\gamma_{k,2}$. Now, pick any path $\beta_i \in B_1$ (B_2). If, for some $j \neq i$, $\gamma_{j,1}$ overlaps (i.e., shares an edge) with $\gamma_{i,1}$, then by the non-reroutability of G , we have

1. $\beta_j \in B_2$ (B_1), in which case $\gamma_{j,2}$ does not overlap with $\gamma_{i,2}$; or

2. $\beta_j \in B_1 (B_2)$, in which case β_j must share

- the edge on $\gamma_{i,1} (\gamma_{i,2})$ ending at $t(\gamma_{i,1}) (t(\gamma_{i,2}))$,
- the subpath $\beta_i[t(\gamma_{i,1}), h(\gamma_{i,2})] (\beta_i[t(\gamma_{i,2}), h(\gamma_{i,1})])$,
- and the edge on $\gamma_{i,2} (\gamma_{i,1})$ starting from $h(\gamma_{i,2}) (h(\gamma_{i,1}))$

with β_i . In the remainder of this proof, we say β_j is in the same *equivalence class* as β_i .

In the following, we say a merging at edge e is ψ -involved if e belongs to either ψ_1 or ψ_2 . The following properties then follow from the non-reroutability of G :

- 1) All B -paths of the same type (meaning all of them belong to either B_1 or B_2) and their equivalent classes can be (partially) ordered in the following sense: Consider $\beta_i, \beta_j \in B$ of the same type. Assume that β_i merges with ψ_1, ψ_2 at $\gamma_{i,1}, \gamma_{i,2}$, and β_j merges with ψ_1, ψ_2 at $\gamma_{j,1}, \gamma_{j,2}$. If $\gamma_{i,1}$ is smaller than $\gamma_{j,1}$, then $\gamma_{i,2}$ must be smaller than $\gamma_{j,2}$; in this case, we say that β_i is *smaller* than β_j , and the equivalence class of β_i is *smaller* than that of β_j . As a result, we can list the equivalence classes of all B_1 -paths in ascending order: Q_1, Q_2, \dots, Q_m , and the equivalence classes of all B_2 -paths in ascending order: $\hat{Q}_1, \hat{Q}_2, \dots, \hat{Q}_n$.
- 2) A merging by two equivalent B -paths or two B -paths of different types must be ψ -involved. If a merging by any two non-equivalent B -paths β_i, β_j is non- ψ -involved, then β_i and β_j are of the same type. If furthermore β_i is smaller than β_j , then there exists u such that $\beta_i \in Q_u(\hat{Q}_u)$ and $\beta_j \in Q_{u+1}(\hat{Q}_{u+1})$. As a consequence, for any u, v

$$|G[Q_u, Q_{u+1}]|_{\mathcal{M}} \leq \min\{|Q_u|, |Q_{u+1}|\}, \quad |G[\hat{Q}_v, \hat{Q}_{v+1}]|_{\mathcal{M}} \leq \min\{|\hat{Q}_v|, |\hat{Q}_{v+1}|\},$$

where $G[Q_u, Q_{u+1}]$ ($G[\hat{Q}_v, \hat{Q}_{v+1}]$) denotes the subgraph of G induced on all the $Q_u(\hat{Q}_v)$ -paths and $Q_{u+1}(\hat{Q}_{v+1})$ -paths.

- 3) Any A -path can merge with at most one B_1 -path and at most one B_2 -path.
- 4) Any three β -paths can only merge with each other at most twice.

Now, by the definition of A and B , we have the number of ψ -involved mergings is upper bounded by

$$|A| + 2|B| = k + |B| - |C| = 2k - |A| - 2|C|,$$

and by Theorem 3.9, the number of non- ψ -involved mergings is upper bounded by $\lfloor \frac{k^2}{4} \rfloor$. It then follows that

$$M(G) \leq k + |B| - |C| + \left\lfloor \frac{k^2}{4} \right\rfloor. \quad (10)$$

Note that for any $1 \leq k \leq 3$,

$$M(G) \leq 2k - |A| - 2|C| + \left\lfloor \frac{k^2}{4} \right\rfloor \leq 2k + \left\lfloor \frac{k^2}{4} \right\rfloor = 3k - 1.$$

So, from now on, we only consider the case when $k \geq 4$. It can be easily checked that when $|B| = k$,

$$M(G) \leq 3k - 1.$$

Next, we show that when $|B| < k$,

$$M(G) \leq \left\lfloor \frac{k^2}{4} \right\rfloor + k + 2.$$

If $|B| - |C| \leq 2$, by (10), the above inequality immediately holds.

If $|B| - |C| \geq 3$, we have the following cases to consider:

Case 1: there exists some equivalence class that has more than one element. Without loss of generality, assume some B_1 -class has more than one element, and let Q_i be the smallest such class with $|Q_i| = m > 1$, and let $Q_{i'}$ be the largest such class with $|Q_{i'}| = m' > 1$.

Case 1.1: the number of non- ψ -involved mergings between Q_i and Q_{i+1} is strictly less than m . Then, one checks that

- $|G[Q_i, \psi]|_{\mathcal{M}} \leq m + 1$, where $G[Q_i, \psi]$ denotes the subgraph of G induced on all the Q_i -paths and ψ -paths;
- the number of ψ -involved mergings by Q_j -paths, $j \neq i$, and ψ -paths is upper bounded by $2(|B| - m) + |A|$.
- the number of non- ψ -involved mergings by Q_i -paths and other B_1 -classes is upper bounded by m .
- By Theorem 3.9, the number of non- ψ -involved mergings among Q_j -paths, $j \neq i$, is upper bounded by $\left\lfloor \frac{(k-m)^2}{4} \right\rfloor$.
- The number of non- ψ -involved mergings by Q_i -paths and (A -paths or C -paths) is upper bounded by $|A| + |C|$.

Combining all the bounds above, we have

$$\begin{aligned} M(G) &\leq (m + 1) + 2(|B| - m) + |A| + m + \left\lfloor \frac{(k - m)^2}{4} \right\rfloor + |A| + |C| \\ &= \left\lfloor \frac{(k - m)^2}{4} \right\rfloor + 2|B| + 2|A| + |C| + 1 \leq \left\lfloor \frac{(k - 2)^2}{4} \right\rfloor + 2k + 1 = \left\lfloor \frac{k^2}{4} \right\rfloor + k + 2. \end{aligned}$$

Case 1.2: the number of non- ψ -involved mergings between Q_i and Q_{i+1} is equal to m , which necessarily implies that $i' \neq i$. Then, for either Q_i or $Q_{i'}$, the number of non- ψ -involved mergings with A -paths is at most $|A| - 1$, so we have

$$\begin{aligned} M(G) &\leq (m + 1) + 2(|B| - m) + A + (m + 1) + \left\lfloor \frac{(k - m)^2}{4} \right\rfloor + (|A| - 1) + |C| \\ &= \left\lfloor \frac{(k - m)^2}{4} \right\rfloor + 2|B| + 2|A| + |C| + 1 \leq \left\lfloor \frac{(k - 2)^2}{4} \right\rfloor + 2k + 1 = \left\lfloor \frac{k^2}{4} \right\rfloor + k + 2. \end{aligned}$$

Case 2: every equivalence class has exactly one element. For this case, since the number of ψ -involved mergings is upper bounded by $|A| + 2|B| = 2k - |A| - 2|C|$, it suffices to show that the number of non- ψ -involved mergings is upper bounded by

$$\left\lfloor \frac{(k-2)^2}{4} \right\rfloor + |A| + 2|C| + 1.$$

Case 2.1: there do not exist non- ψ -involved mergings among all equivalence classes. For this case, the total number of mergings by $\{\beta_i, \beta_j\}$, any two chosen B -paths of the same type, and (A -paths or C -paths) is at most $|A| + 2|C|$. We then conclude that the number of non- ψ -involved mergings is upper bounded by

$$\left\lfloor \frac{(k-2)^2}{4} \right\rfloor + |A| + 2|C|.$$

Case 2.2: there exists a non- ψ -involved merging by two adjacent equivalent classes, say Q_j, Q_{j+1} , and these two classes merge with each other once, however they do not merge with any other B_1 -classes. By Property 2), both of these two classes are of the same type. Moreover, by Property 3), the number of mergings between these two classes and (A -paths or C -paths) is at most $|A| + 2|C|$. Hence, we have the number of non- ψ -involved mergings is upper bounded by

$$\left\lfloor \frac{(k-2)^2}{4} \right\rfloor + |A| + 2|C| + 1.$$

Case 2.3: there exist at least three adjacent equivalent classes, say $Q_j, Q_{j+1}, \dots, Q_{j+l}$, $l \geq 2$, such that Q_{j+r} merges with Q_{j+r+1} , $r = 0, 1, \dots, l-1$, however there are no mergings by $\{Q_j, Q_{j+1}, \dots, Q_{j+l}\}$ and other B_1 -classes. For this case, it can be checked that at least one of $\{Q_j, Q_{j+1}\}$, $\{Q_{j+l-1}, Q_{j+l}\}$ and $\{Q_j, Q_{j+l}\}$ merges with A -paths at most $|A| - 1$ times. Since each of the above pair of paths merge with B -paths at most twice and merges with C -paths at most $2|C|$ times, we thus have the number of non- ψ -involved mergings is upper bounded by

$$\left\lfloor \frac{(k-2)^2}{4} \right\rfloor + (|A| - 1) + 2 + 2|C| = \left\lfloor \frac{(k-2)^2}{4} \right\rfloor + |A| + 2|C| + 1.$$

Now, combining all the cases above, we then have established the upper bound direction:

$$\mathcal{M}(\underbrace{1, \dots, 1}_k, 2) \leq \max\{3k - 1, \left\lfloor \frac{k^2}{4} \right\rfloor + k + 2\} = \begin{cases} 3k - 1 & \text{if } 4 \leq k \leq 6, \\ \left\lfloor \frac{k^2}{4} \right\rfloor + k + 2 & \text{if } k > 6. \end{cases}$$

The lower bound direction: First, consider the following $(\underbrace{1, 1, \dots, 1}_k, 2)$ -graph G with distinct

sources $S_1, S_2, \dots, S_k, \widehat{S}$, sinks $R_1, R_2, \dots, R_k, \widehat{R}$, a Menger's path β_i from S_i to R_i for $1 \leq i \leq k$, two Menger's paths ψ_1, ψ_2 from \widehat{S} to \widehat{R} such that

- every merging in G is by exactly two paths;
- for $i = 1, 2, \dots, k$, β_i first merges with ψ_1 at $\lambda_{i,1}$, and then merges with ψ_2 at $\lambda_{i,2}$;

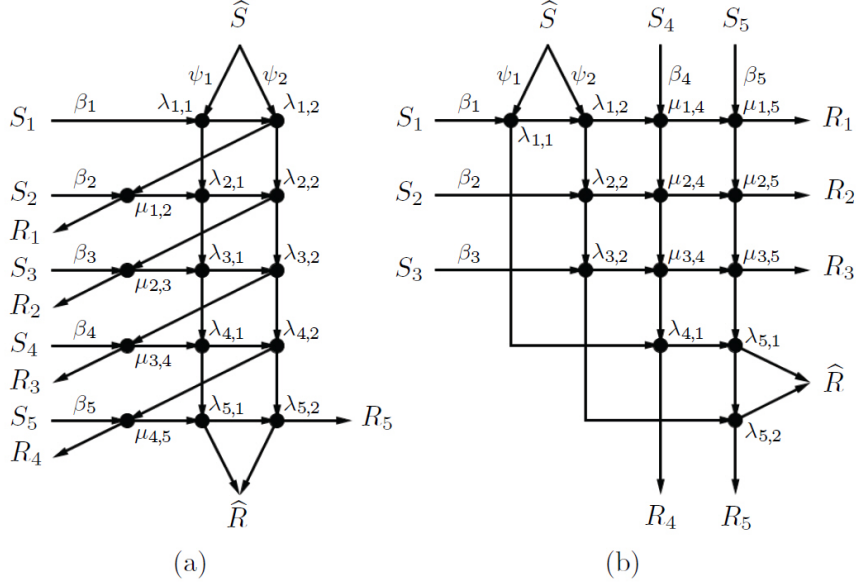


Figure 12: (a) A non-reroutable $(1,1,1,1,2)$ -graph with 14 mergings (b) A non-reroutable $(1,1,1,1,2)$ -graph with 13 mergings

- for any $i < j$, β_i is smaller than β_j ;
- for any $i = 1, 2, \dots, k-1$, β_i merges with β_{i+1} at $\mu_{i,i+1}$ such that $\mu_{i,i+1}$ is larger than $\lambda_{i,2}$ and smaller than $\lambda_{i+1,1}$;
- there are no other mergings.

See Figure 12(a) for an example. It can be verified that the above G is a non-reroutable $(\underbrace{1, 1, \dots, 1}_k, 2)$ -graph with $3k - 1$ mergings, which implies that

$$\mathcal{M}(\underbrace{1, 1, \dots, 1}_k, 2) \geq 3k - 1. \quad (11)$$

Next, consider the following $(\underbrace{1, 1, \dots, 1}_k, 2)$ -graph G with distinct sources $S_1, S_2, \dots, S_k, \widehat{S}$, sinks $R_1, R_2, \dots, R_k, \widehat{R}$, a Menger's path β_i from S_i to R_i for $1 \leq i \leq k$, two Menger's paths ψ_1, ψ_2 from \widehat{S} to \widehat{R} such that

- every merging in G is by exactly two paths;
- for $i = 1, 2, \dots, \lceil k/2 \rceil$, $j = \lceil k/2 \rceil + 1, \dots, k$, β_i merges with β_j at $\mu_{i,j}$;
- for any $i = 1, 2, \dots, \lceil k/2 \rceil$ and any $\lceil k/2 \rceil + 1 \leq j_1 < j_2 \leq k$, μ_{i,j_1} is smaller than μ_{i,j_2} ;
- for any $j = \lceil k/2 \rceil + 1, \dots, k$ and any $1 \leq i_1 < i_2 \leq \lceil k/2 \rceil$, $\mu_{i_1,j}$ is smaller than $\mu_{i_2,j}$;
- for $j = \lceil k/2 \rceil + 1, \dots, k$, ψ_1 merges with β_j at $\lambda_{j,1}$ such that $\lambda_{j,1}$ is larger than $\mu_{\lceil k/2 \rceil, j}$;

- for $i = 1, 2, \dots, \lceil k/2 \rceil$, ψ_2 merges with β_i at $\lambda_{i,2}$ such that $\lambda_{i,2}$ is smaller than $\mu_{i,\lceil k/2 \rceil+1}$;
- ψ_1 merges with β_1 at $\lambda_{1,1}$ such that $\lambda_{1,1}$ is smaller than $\lambda_{1,2}$ and $\lambda_{\lceil k/2 \rceil+1,1}$;
- ψ_2 merges with β_k at $\lambda_{k,2}$ such that $\lambda_{k,2}$ is larger than $\lambda_{\lceil k/2 \rceil,2}$ and $\lambda_{k,1}$;
- there are no other mergings.

See Figure 12(b) for an example. It can be verified that the above G is a non-reroutable $(\underbrace{1, 1, \dots, 1}_k, 2)$ -graph with $\lfloor k^2/4 \rfloor + k + 2$ mergings, which implies that

$$\mathcal{M}(\underbrace{1, 1, \dots, 1}_k, 2) \geq \left\lfloor \frac{k^2}{4} \right\rfloor + k + 2. \quad (12)$$

Combining (11) and (12), we then have established the lower bound direction. \square

Theorem 3.12.

$$\mathcal{M}(\underbrace{1, 1, \dots, 1}_k, n) = nk + \left\lfloor \frac{k^2}{4} \right\rfloor \quad \text{for } n \geq \frac{3k-1}{4}.$$

Proof. For the “ \leq ” direction, it follows from [5] that

$$\mathcal{M}(\underbrace{1, 1, \dots, 1}_k, n) \leq \mathcal{M}(\underbrace{1, 1, \dots, 1}_k) + k\mathcal{M}(1, n) = nk + \left\lfloor \frac{k^2}{4} \right\rfloor.$$

Next, we show that the following $(\underbrace{1, 1, \dots, 1}_k, n)$ -graph G , which has distinct sources $S_1, S_2, \dots, S_k, \widehat{S}$ and sinks $R_1, R_2, \dots, R_k, \widehat{R}$, is non-reroutable with $nk + \lfloor \frac{k^2}{4} \rfloor$ mergings. The graph G (see Figure 13 for an example) can be described as follows:

- There is a path β_i from S_i to R_i for $1 \leq i \leq k$ and n Menger’s paths $\Psi = \{\psi_1, \psi_2, \dots, \psi_k\}$ from \widehat{S} to \widehat{R} ;
- For any feasible i, j , β_i merges with ψ_j exactly once at the merging $\lambda_{i,j}$;
- For any feasible i, j , β_i in B_1 or B_3 merges with β_j in B_2 exactly once at the merging $\mu_{i,j}$, where

$$\begin{aligned} B_1 &= \{\beta_1, \beta_2, \dots, \beta_{k_1}\}, \\ B_2 &= \{\beta_{k_1+1}, \beta_{k_1+2}, \dots, \beta_{k_1+k_2}\}, \\ B_3 &= \{\beta_{k_1+k_2+1}, \beta_{k_1+k_2+2}, \dots, \beta_k\}, \end{aligned}$$

here, $k_1 = \lceil \lceil k/2 \rceil / 2 \rceil$, $k_2 = \lfloor k/2 \rfloor$, $k_3 = \lfloor \lceil k/2 \rceil / 2 \rfloor$;

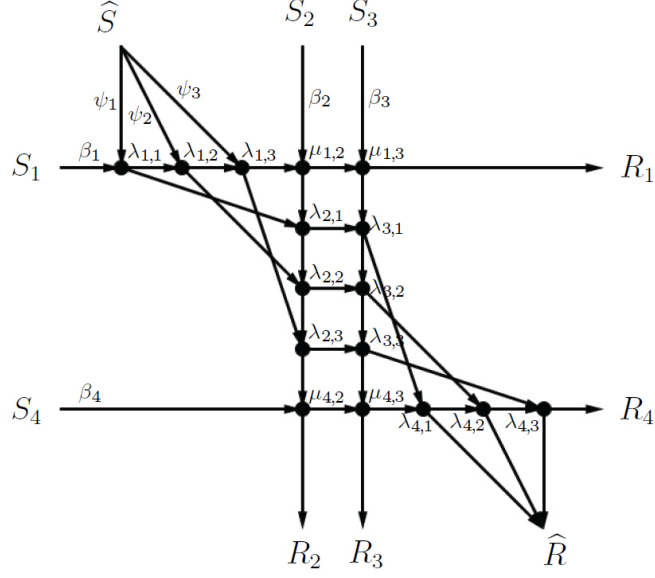


Figure 13: A non-reroutable $(1, 1, 1, 1, 3)$ -graph with 16 mergings

- The mergings on each path can be sequentially listed in the ascending order as follows:
for $1 \leq i \leq k$,

$$\psi_i : \lambda_{1,i}, \lambda_{2,i}, \dots, \lambda_{n,i};$$

for $1 \leq i \leq k_1$,

$$\beta_i : \lambda_{i,1}, \lambda_{i,2}, \dots, \lambda_{i,k}, \mu_{i,k_1+1}, \mu_{i,k_1+2}, \dots, \mu_{i,k_1+k_2};$$

for $k_1 + 1 \leq i \leq k_1 + k_2$,

$$\beta_i : \mu_{1,i}, \mu_{2,i}, \dots, \mu_{k_1,i}, \lambda_{i,1}, \lambda_{i,2}, \dots, \lambda_{i,k}, \mu_{k_1+k_2+1,i}, \mu_{k_1+k_2+2,i}, \dots, \mu_{k,i};$$

for $k_1 + k_2 + 1 \leq i \leq k$,

$$\beta_i : \mu_{i,k_1+1}, \mu_{i,k_1+2}, \dots, \mu_{i,k_1+k_2}, \lambda_{i,1}, \lambda_{i,2}, \dots, \lambda_{i,k}.$$

It can be checked that G is non-reroutable with

$$\begin{aligned} |G|_{\mathcal{M}} &= n(k_1 + k_2 + k_3) + (k_1 + k_3)k_2 \\ &= n \left(\left\lceil \frac{\lceil \frac{k}{2} \rceil}{2} \right\rceil + \left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{\lceil \frac{k}{2} \rceil}{2} \right\rfloor \right) + \left(\left\lceil \frac{\lceil \frac{k}{2} \rceil}{2} \right\rceil + \left\lfloor \frac{\lceil \frac{k}{2} \rceil}{2} \right\rfloor \right) \left\lfloor \frac{k}{2} \right\rfloor \\ &= nk + \left\lfloor \frac{k^2}{4} \right\rfloor. \end{aligned}$$

□

4 Bounds

4.1 Bounds on $\mathcal{M}^*(n, n)$

In this section, we will construct a non-reroutable (n, n) -graph $\mathcal{E}(n, n)$ with one source S , two sinks R_1, R_2 , a set of Menger's paths $\phi = \{\phi_0, \phi_1, \dots, \phi_{n-1}\}$ from S to R_1 , a set of Menger's paths $\psi = \{\psi_0, \psi_1, \dots, \psi_{n-1}\}$ from S to R_2 and $(n-1)^2$ mergings for any positive integer n , thus giving a lower bound on $\mathcal{M}^*(n, n)$.

The graph $\mathcal{E}(n, n)$ can be described as follows: for each $0 \leq i \leq n-1$, paths ϕ_i and ψ_i share a starting subpath ω_i . After ω_{n-1} , path ϕ_{n-1} does not merge any more, directly "flowing" to R_1 ; after ω_0 , path ψ_0 does not merge any more, directly "flowing" to R_2 . The rest of the graph can be determined how paths $\phi_0, \phi_1, \dots, \phi_{n-2}$ merge with $\psi_1, \psi_2, \dots, \psi_{n-1}$. In more detail, for a given n , we define

$$X = \{x_{i,j} = i(2n - i - 2) + j : 0 \leq i \leq n-2, 1 \leq j \leq n - i - 1\}$$

and

$$Y = \{y_{i,j} = i(2n - i - 3) + (n-1) + j : 0 \leq i \leq n-3, 1 \leq j \leq n - i - 2\}.$$

It can be checked that all $x_{i,j}$'s, $y_{i,j}$'s are distinct and

$$X \cup Y = \{1, 2, \dots, (n-1)^2\}.$$

Now we define a mapping $f : \{1, 2, \dots, (n-1)^2\} \mapsto \{(i, j) : 0 \leq i, j \leq n-1\}$ by

$$f(k) = \begin{cases} (i, j) & \text{if } k = x_{i,j}, \\ (n-1-j, n-1-i) & \text{if } k = y_{i,j}. \end{cases}$$

Then the merging sequence of the rest of the graph can be defined as

$$\Omega = [\Omega_k : \Omega_k = f(k), 1 \leq k \leq (n-1)^2].$$

For example, $\mathcal{E}(4, 4)$, as illustrated in Figure 14, is determined by the merging sequence

$$\Omega = [(0, 1), (0, 2), (0, 3), (2, 3), (1, 3), (1, 1), (1, 2), (2, 2), (2, 1)].$$

Now, we prove that

Lemma 4.1. $\mathcal{E}(n, n)$ is non-reroutable.

Proof. Let $z = n-1$. For each $i, j = 0, 1, \dots, z$, label each merging (i, j) in the merging sequence as $\gamma_{i,j}$ (it can be easily checked that no two mergings share the same label).

We only prove that there is only one possible set of Menger's paths from S to R_1 . The uniqueness of Menger's path sets from S to R_2 can be established using a parallel argument.

Let α_1 be an arbitrary yet fixed set of Menger's paths from S to R_1 . It suffices to prove that α_1 is non-reroutable. Note that each path in α_1 must end with either $\omega_z \rightarrow R_1$ or $\gamma_{i,z-i} \rightarrow R_1$, $i = 0, 1, \dots, z-1$ (here and hereafter, slightly abusing the notations " \rightarrow " and " \leftarrow ", for paths (or vertices) A_1, A_2, \dots, A_k , we use $A_1 \rightarrow A_2 \rightarrow \dots \rightarrow A_k$ or $A_k \leftarrow \dots \leftarrow A_1$).

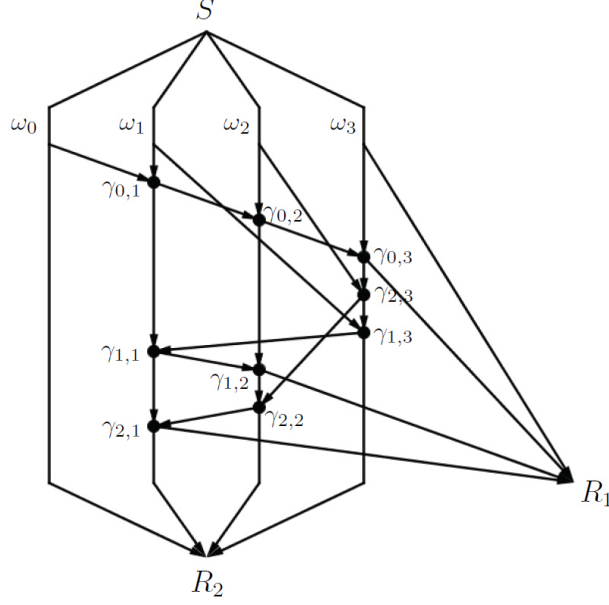


Figure 14: Graph $\mathcal{E}(4, 4)$ with 9 mergings

$A_2 \leftarrow A_1$ to denote the path which sequentially passes through A_1, A_2, \dots, A_k ; it can be checked that in this proof such an expression uniquely determines a path). In α_1 , label the Menger's path ending with $\gamma_{i,z-i} \rightarrow R_1$ as the i -th Menger's path for $0 \leq i \leq z-1$, and the Menger's path ending with $\omega_z \rightarrow R_1$ as the z -th one.

It is obvious that in $\mathcal{E}(m, m)$, there is only one path ending with $\omega_z \rightarrow R_1$, which implies that the z -th Menger's path in α_1 is "fixed" (as $S \rightarrow \omega_z \rightarrow R_1$); or, more rigorously, for any set of Menger's paths α'_1 , the z -th Menger's path in α'_1 is the same as the z -th one in α_1 . So, for the purpose of choosing other Menger's paths, all the edges on $S \rightarrow \omega_z \rightarrow R_1$ are "occupied". It then follows that, in α_1 , $\gamma_{0,z}$ must "come" from $\gamma_{0,z-1}$; more precisely, in α_1 , $\gamma_{0,z-1}$ is smaller than $\gamma_{0,z}$ on the 0-th path and there is no other merging between them on this path. Now, all the edges on $\gamma_{0,z-1} \rightarrow \gamma_{0,z} \rightarrow R_1$ are occupied.

Inductively, only considering unoccupied edges, one can check that for $0 \leq i \leq z-2$, $\gamma_{i,z-i}$ must come from $\gamma_{i,z-i-1}$; in other words, for $0 \leq i \leq z-2$, the i -th Menger's path must end with $\gamma_{i,z-i-1} \rightarrow \gamma_{i,z-i} \rightarrow R_1$. It then follows that the $(z-1)$ -th Menger's path must come from $\gamma_{z-1,2} \leftarrow \gamma_{z-1,3} \leftarrow \dots \leftarrow \gamma_{z-1,z} \leftarrow \omega_{z-1}$; so, the $(z-1)$ -th Menger's path is fixed as $S \rightarrow \omega_{z-1} \rightarrow \gamma_{z-1,z} \rightarrow \gamma_{z-1,z-1} \rightarrow \dots \rightarrow \gamma_{z-1,2} \rightarrow \gamma_{z-1,1} \rightarrow R_1$.

We now proceed by induction on j , $j = z-2, z-3, \dots, 1$. Suppose that, for $j+1 \leq i \leq z$, the i -th Menger's path is already fixed (and hence the edges on these paths are all occupied), and for $0 \leq i \leq j$, the i -th Menger's path ends with $\gamma_{i,j-i+1} \rightarrow \gamma_{i,j-i+2} \rightarrow \dots \rightarrow \gamma_{i,z-i} \rightarrow R_1$ (so, the edges on these paths are all occupied). Only considering the unoccupied edges, one checks that for $0 \leq i \leq j-1$, $\gamma_{i,j-i+1}$ must come from $\gamma_{i,j-i}$. It then follows that the j -th Menger's path, which ends with $\gamma_{j,1} \rightarrow \gamma_{j,2} \rightarrow \dots \rightarrow \gamma_{j,z-j} \rightarrow R_1$, must come from $\gamma_{j,z-j+1} \leftarrow \gamma_{j,z-j+2} \leftarrow \dots \leftarrow \gamma_{j,z} \leftarrow \omega_j$. So, the j -th Menger's path can now be fixed as $S \rightarrow \omega_j \rightarrow \gamma_{j,z} \rightarrow \gamma_{j,z-1} \rightarrow \dots \rightarrow \gamma_{j,z-j+1} \rightarrow \gamma_{j,1} \rightarrow \gamma_{j,2} \rightarrow \dots \rightarrow \gamma_{j,z-j} \rightarrow R_1$. Now, for $j \leq i \leq z$, the i -th Menger's path is fixed, and for $0 \leq i \leq j-1$, the i -th Menger's path must end with $\gamma_{i,j-i} \rightarrow \gamma_{i,j-i+1} \rightarrow \dots \rightarrow \gamma_{i,z-i} \rightarrow R_1$.

It follows from the above inductive argument that for $1 \leq i \leq z$, the i -th Menger's path is fixed, and the 0-th Menger's path must end with $\gamma_{0,1} \rightarrow \gamma_{0,2} \rightarrow \cdots \rightarrow \gamma_{0,z} \rightarrow R_1$. One then checks that the $\gamma_{0,1}$ must come from ω_0 , which implies that the 0-th Menger's path is fixed as $S \rightarrow \omega_0 \rightarrow \gamma_{0,1} \rightarrow \gamma_{0,2} \rightarrow \cdots \rightarrow \gamma_{0,z} \rightarrow R_1$. The proof of uniqueness of Menger's path set from S to R_1 is then complete. \square

The above lemma then immediately implies that

Theorem 4.2.

$$\mathcal{M}^*(n, n) \geq (n-1)^2.$$

The following theorem gives an upper bound on $\mathcal{M}^*(n, n)$. First, we remind the reader that, by Proposition 3.6 in [5], $\mathcal{M}^*(m, n) = \mathcal{M}^*(n, n)$ for any $m \geq n$.

Theorem 4.3.

$$\mathcal{M}^*(n, n) \leq \left\lceil \frac{n}{2} \right\rceil (n^2 - 4n + 5).$$

Proof. Consider any (n, n) -graph G with one source S , sinks R_1, R_2 , a set of Menger's paths $\phi = \{\phi_1, \phi_2, \dots, \phi_n\}$ from S to R_1 , a set of Menger's paths $\psi = \{\psi_1, \psi_2, \dots, \psi_n\}$ from S to R_2 .

As discussed in Section 2.2, we assume that, for $1 \leq i \leq n$, paths ϕ_i and ψ_i share a starting subpath, and paths ϕ_n and ψ_0 do not merge with any other paths, directly flowing to the sinks (then, necessarily, each ψ -AA-sequence is of positive length, and by Lemma 2.4, the shortest ψ -AA-sequence is of length 1). We say that the path pair (ϕ_i, ψ_j) is *matched* if $i = j$, otherwise, *unmatched*. Apparently, each starting subpath corresponds to a matched path pair; and among the set of all path pairs, each of which corresponds some merging in G , there are at most $(n-2)$ matched and at most $(n^2 - 3n + 3)$ unmatched.

We then consider the following two cases (note that the following two cases may not be mutually exclusive):

Case 1: there exists a shortest ψ -AA-sequence associated with a matched path pair. By Lemma 2.5 and the fact that each starting subpath corresponds to a matched path pair, there are at most $\left\lfloor \frac{n-1}{2} \right\rfloor$ mergings corresponding to this path pair, at most $\left\lfloor \frac{n-2}{2} \right\rfloor$ corresponding to any other matched path pair, and at most $\left\lfloor \frac{n-1}{2} \right\rfloor$ mergings corresponding to any unmatched. So, the number of mergings is upper bounded by

$$\left\lfloor \frac{n-1}{2} \right\rfloor + (n-3) \left\lfloor \frac{n-2}{2} \right\rfloor + (n^2 - 3n + 3) \left\lfloor \frac{n-1}{2} \right\rfloor. \quad (13)$$

Case 2: there exists a shortest ψ -AA-sequence associated with an unmatched path pair. Again, by Lemma 2.5 and the fact that each starting subpath corresponds to a matched path pair, there are at most $\left\lfloor \frac{n}{2} \right\rfloor$ mergings corresponding to this path pair, at most $\left\lfloor \frac{n-1}{2} \right\rfloor$ mergings corresponding to any other unmatched path pair, and at most $\left\lfloor \frac{n-2}{2} \right\rfloor$ mergings corresponding to any matched. So, the number of mergings is upper bounded by

$$\left\lfloor \frac{n}{2} \right\rfloor + (n-2) \left\lfloor \frac{n-2}{2} \right\rfloor + (n^2 - 3n + 2) \left\lfloor \frac{n-1}{2} \right\rfloor. \quad (14)$$

Then $\mathcal{M}^*(n, n) \leq \max\{(13), (14)\}$. For odd n , (13) is larger than (14), so we have

$$\begin{aligned}\mathcal{M}^*(n, n) &\leq \binom{\frac{n-1}{2}}{2} + (n-3) \binom{\frac{n-3}{2}}{2} + (n^2 - 3n + 3) \binom{\frac{n-1}{2}}{2} \\ &= (n^2 - 4n + 5) \binom{\frac{n+1}{2}}{2}.\end{aligned}$$

For even n , (14) is larger than (13), so we have

$$\begin{aligned}\mathcal{M}^*(n, n) &\leq \binom{\frac{n}{2}}{2} + (n-2) \binom{\frac{n-2}{2}}{2} + (n^2 - 3n + 2) \binom{\frac{n-2}{2}}{2} \\ &= (n^2 - 4n + 5) \binom{\frac{n}{2}}{2}.\end{aligned}$$

The proof is then complete. \square

4.2 Bounds on $\mathcal{M}(m, n)$

Consider the following (n, n) -graph $\mathcal{F}(n, n)$ with distinct sources S_1, S_2 , sinks R_1, R_2 , a set of Menger's paths $\phi = \{\phi_1, \phi_2, \dots, \phi_n\}$ from S_1 to R_1 , a set of Menger's paths $\psi = \{\psi_1, \psi_2, \dots, \psi_n\}$ from S_2 to R_2 , and a merging sequence $\Omega = [\Omega_k : 1 \leq k \leq 2n^2 - 3n + 2]$, where

$$\Omega_k = \begin{cases} ([j - i]_n, i + 1) & \text{if } k = 2i(n - 1) + j \\ & \text{for } (0 \leq i \leq n - 1, 1 \leq j \leq n - 1) \text{ or } (i = n - 1, j = n), \\ (n - i, [i - j + 2]_n) & \text{if } k = (2i + 1)(n - 1) + j \text{ for } 0 \leq i \leq n - 2, 1 \leq j \leq n - 1, \end{cases}$$

where, for any integer x , $[x]_n$ denotes the least strictly positive residue of x modulo n . For a quick example, see $\mathcal{F}(3, 3)$ in Figure 15(a), whose merging sequence is

$$\Omega = [(1, 1), (2, 1), (3, 1), (3, 3), (3, 2), (1, 2), (2, 2), (2, 1), (2, 3), (3, 3), (1, 3)].$$

Then, similar to the proof of Lemma 4.1, through verifying the uniqueness of the set of Menger's paths from S_i to R_i , we can show that

Lemma 4.4. $\mathcal{F}(n, n)$ is non-reroutable.

Consider a non-reroutable (k, n) -graph $\mathcal{G}(k, n)$ with distinct sources $\widehat{S}_1, \widehat{S}_2$, sinks $\widehat{R}_1, \widehat{R}_2$, a set of Menger's paths $\widehat{\phi} = \{\widehat{\phi}_1, \widehat{\phi}_2, \dots, \widehat{\phi}_k\}$ from \widehat{S}_1 to \widehat{R}_1 , a set of Menger's paths $\widehat{\psi} = \{\widehat{\psi}_1, \widehat{\psi}_2, \dots, \widehat{\psi}_n\}$ from \widehat{S}_2 to \widehat{R}_2 . For a fixed merging sequence of $\mathcal{G}(k, n)$, assume, without loss of generality, that the first element is $(\widehat{\phi}_1, \widehat{\psi}_n)$. Now, we consider the following procedure of concatenating graphs $\mathcal{F}(n, n)$ and $\mathcal{G}(k, n)$ to obtain a new graph:

1. split R_1 into n copies $R_1^{(1)}, R_1^{(2)}, \dots, R_1^{(n)}$ such that path ϕ_i has the ending point $R_1^{(i)}$;
split R_2 into n copies $R_2^{(1)}, R_2^{(2)}, \dots, R_2^{(n)}$ such that path ψ_i has the ending point $R_2^{(i)}$;
2. split \widehat{S}_1 into k copies $\widehat{S}_1^{(1)}, \widehat{S}_1^{(2)}, \dots, \widehat{S}_1^{(k)}$ such that path $\widehat{\phi}_i$ has the starting point $\widehat{S}_1^{(i)}$;
split \widehat{S}_2 into n copies $\widehat{S}_2^{(1)}, \widehat{S}_2^{(2)}, \dots, \widehat{S}_2^{(n)}$ such that path $\widehat{\psi}_i$ has the starting point $\widehat{S}_2^{(i)}$;

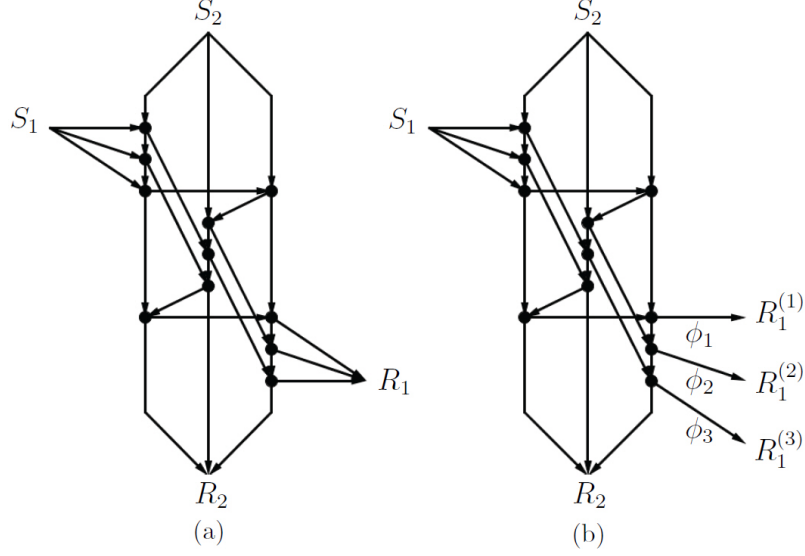


Figure 15: (a) Graph $\mathcal{F}(3,3)$ with 11 mergings (b) Splitting of R_1 in $\mathcal{F}(3,3)$

3. delete all edges on ϕ_1 and all edges on ψ_n , each of which is larger than merging (ϕ_1, ψ_n) to obtain new ϕ_1 and ψ_n ;
4. delete all edges on $\hat{\phi}_1$ and all edges on $\hat{\psi}_n$, each of which is smaller than merging $(\hat{\phi}_1, \hat{\psi}_n)$ to obtain new $\hat{\phi}_1$ and $\hat{\psi}_n$;
5. concatenate ϕ_1 and $\hat{\phi}_1$ to obtain $\phi_1 \circ \hat{\phi}_1$ (so, necessarily, ψ_n and $\hat{\psi}_n$ are concatenated simultaneously and we obtain $\psi_n \circ \hat{\psi}_n$);
6. identify $S_1, \hat{S}_1^{(2)}, \hat{S}_1^{(3)}, \dots, \hat{S}_1^{(k)}$; identify $\hat{R}_1, R_1^{(2)}, R_1^{(3)}, \dots, R_1^{(k)}$; identify $R_2^{(i)}$ and $\hat{S}_2^{(i)}$ for $1 \leq i \leq n-1$.

Obviously, such procedure produces a $(k+n-1, n)$ -graph with two distinct sources S_1, S_2 and two sinks \hat{R}_1 and \hat{R}_2 , a set of Menger's paths $\{\phi_1 \circ \hat{\phi}_1, \phi_2, \phi_3, \dots, \phi_n, \hat{\phi}_2, \hat{\phi}_3, \dots, \hat{\phi}_k\}$ from S_1 to \hat{R}_1 and a set of Menger's paths $\{\psi_1 \circ \hat{\psi}_1, \psi_2 \circ \hat{\psi}_2, \dots, \psi_n \circ \hat{\psi}_n\}$ from S_2 to \hat{R}_2 .

For example, in Figure 16, we concatenate $\mathcal{F}(2,2)$ and a non-reroutable $(2,2)$ -graph to obtain a $(3,2)$ -graph. We have the following lemma, whose proof is similar to Lemma 4.1 and thus omitted.

Lemma 4.5. *The concatenated graph as above is a non-reroutable $(k+n-1, n)$ -graph with the number of mergings equal to $|\mathcal{F}(n, n)|_{\mathcal{M}} + |\mathcal{G}(k, n)|_{\mathcal{M}} - 1$.*

We are now ready for the following theorem, which gives us a lower bound on $\mathcal{M}(m, n)$.

Theorem 4.6.

$$\mathcal{M}(m, n) \geq 2mn - m - n + 1.$$

Proof. Without loss of generality, assume that $m \leq n$. For $1 \leq m' \leq m$ and $1 \leq n' \leq n$, we will iteratively construct a sequence of non-reroutable (m', n') -graphs with $2m'n' - m' - n' + 1$ mergings, which immediately implies the theorem.

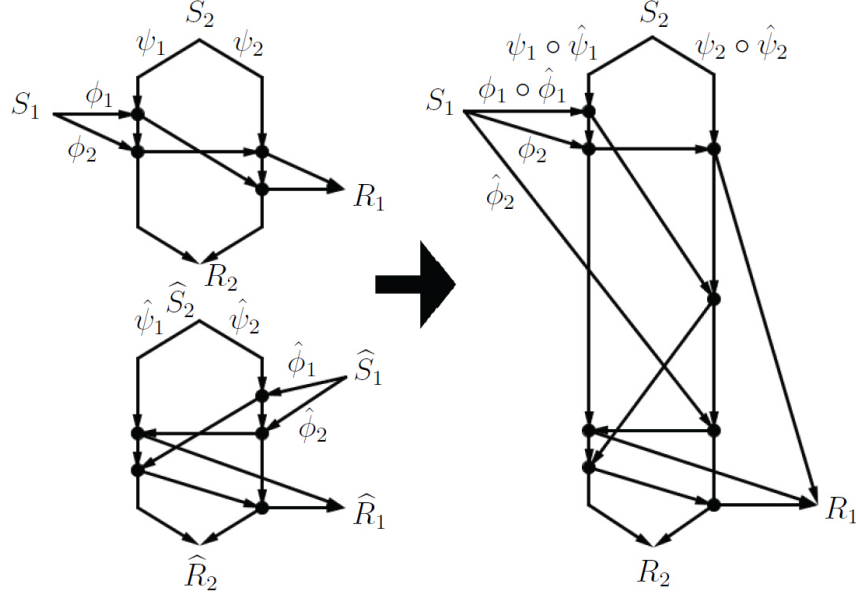


Figure 16: Concatenation of $\mathcal{F}(2, 2)$ and a non-reroutable $(2, 2)$ -graph

First, for any k , $\mathcal{H}(1, k)$, a non-reroutable $(1, k)$ -graph can be given by specifying its mergings sequence

$$\Omega = [(1, 1), (1, 2), \dots, (1, k)].$$

Next, consider the case $2 \leq m \leq n$. Assume that for any m', n' such that $m' \leq n'$, $m' \leq m$, $n' \leq n$, however $(m', n') \neq (m, n)$, we have constructed a non-reroutable (m', n') -graph, which is effectively a non-reroutable (n', m') -graph as well. We obtain a new (m, n) -graph through the following procedure:

1. if $m = n$, concatenate $\mathcal{F}(m, m)$ and an already constructed non-reroutable $(1, m)$ -graph $\mathcal{H}(1, m)$;
2. if $m < n$, concatenate $\mathcal{F}(m, m)$ and an already constructed non-reroutable $(n - m + 1, m)$ -graph.

For the first case, according to Lemma 4.5, the obtained graph is non-reroutable (m, m) -graph with the number of mergings

$$(2m^2 - 3m + 2) + m - 1 = 2m^2 - 2m + 1.$$

Similarly, for the second case, the obtained graph is a non-reroutable (m, n) -graph with the number of mergings

$$(2m^2 - 3m + 2) + (2(n - m + 1)m - (n - m + 1) - m + 1) - 1 = 2mn - m - n + 1.$$

We then have established the theorem. \square

Example 4.7. To construct a non-reroutable $(4, 6)$ -graph with 39 mergings, one can concatenate $\mathcal{F}(4, 4)$ and a non-reroutable $(3, 4)$ -graph, which can be obtained by concatenating $\mathcal{F}(3, 3)$ and a non-reroutable $(2, 3)$ -graph. The latter can be obtained by concatenating $\mathcal{F}(2, 2)$ and a non-reroutable $(2, 2)$ -graph. Finally, a non-reroutable $(2, 2)$ -graph can be obtained by concatenating $\mathcal{F}(2, 2)$ and $\mathcal{H}(1, 2)$. One readily checks that the number of mergings in the eventually obtained graph is

$$|\mathcal{F}(4, 4)|_{\mathcal{M}} + |\mathcal{F}(3, 3)|_{\mathcal{M}} + |\mathcal{F}(2, 2)|_{\mathcal{M}} + |\mathcal{F}(2, 2)|_{\mathcal{M}} + |\mathcal{H}(1, 2)|_{\mathcal{M}} - 4 = 22 + 11 + 4 + 4 + 2 - 4 = 39.$$

Theorem 4.8.

$$\mathcal{M}(m, n) \leq (m + n - 1) + (mn - 2) \left\lfloor \frac{m + n - 2}{2} \right\rfloor.$$

Proof. Consider any (m, n) -graph G with distinct sources S_1, S_2 , sinks R_1, R_2 , a set of Menger's paths $\phi = \{\phi_1, \phi_2, \dots, \phi_m\}$ from S_1 to R_1 , a set of Menger's paths $\psi = \{\psi_1, \psi_2, \dots, \psi_n\}$ from S_2 to R_2 . As discussed in Section 2.2, we assume that all the AA-sequences are of positive lengths. By Lemma 2.4, the shortest ϕ -AA-sequence and ψ -AA-sequence are both of length 1. We then consider the following two cases (note that the following two cases may not be mutually exclusive):

Case 1: there exists a shortest ϕ -AA-sequence and a shortest ψ -AA-sequence, which are associated with the same path pair. By Lemma 2.5, there are at most $\left\lfloor \frac{m+n}{2} \right\rfloor$ mergings corresponding to this path pair, and at most $\left\lfloor \frac{m+n-2}{2} \right\rfloor$ mergings corresponding to any other path pair. So, the number of mergings is upper bounded by

$$\left\lfloor \frac{m+n}{2} \right\rfloor + (mn - 1) \left\lfloor \frac{m+n-2}{2} \right\rfloor. \quad (15)$$

Case 2: there exists a shortest ϕ -AA-sequence and a shortest ψ -AA-sequence, which are associated with two distinct path pairs. Again, by Lemma 2.5, there are at most $\left\lfloor \frac{m+n-1}{2} \right\rfloor$ mergings corresponding to each of these two path pairs, and at most $\left\lfloor \frac{m+n-2}{2} \right\rfloor$ mergings corresponding to any other path pair. So, the number of mergings is upper bounded by

$$2 \left\lfloor \frac{m+n-1}{2} \right\rfloor + (mn - 2) \left\lfloor \frac{m+n-2}{2} \right\rfloor. \quad (16)$$

Then, $\mathcal{M}(m, n) \leq \max\{(15), (16)\}$. Straightforward computations then lead to the theorem. \square

Remark 4.9. It has been established in [6] that

$$n(n-1)/2 \leq \mathcal{M}^*(n, n) \leq n^3.$$

Summarizing all the four bounds we obtain, we have

$$(n-1)^2 \leq \mathcal{M}^*(n, n) \leq \left\lceil \frac{n}{2} \right\rceil (n^2 - 4n + 5),$$

$$2mn - m - n + 1 \leq \mathcal{M}(m, n) \leq (m + n - 1) + (mn - 2) \left\lfloor \frac{m + n - 2}{2} \right\rfloor.$$

4.3 Bounds on $\mathcal{M}(3, n)$

It has been shown in [5] that for any k , there exists C_k such that $\mathcal{M}(k, n) \leq C_k n$ for all n , where C_k can be rather loose. The following result refines the above result for the case when $k = 3$.

Theorem 4.10.

$$\mathcal{M}(3, n) \leq 14n.$$

Proof. Consider any non-reroutable $(3, n)$ -graph G with distinct sources S_1, S_2 , sinks R_1, R_2 , a set of Menger's paths $\phi = \{\phi_1, \phi_2, \phi_3\}$ from S_1 to R_1 and a set of Menger's paths $\psi = \{\psi_1, \psi_2, \dots, \psi_n\}$ from S_2 to R_2 . If a merging is the smallest (the largest) one on a ψ -path, we say it is an x -terminal (y -terminal) merging on the ψ -path, or simply a ψ -terminal merging.

Consider the following iterative procedure (Figures 17, 18 and 19 roughly illustrate the procedure), where, for notational simplicity, we treat a graph as a union of its vertex set and edge set. Initially set $\mathbb{S}^{(0)} = \emptyset$, and $\mathbb{R}^{(0)} = G$. Now for each $j = 1, 2, 3$, pick a merging $\gamma_{0,j}$ such that $\gamma_{0,j}$ belongs to path ϕ_j and

$$|\mathbb{R}^{(0)}|t(\gamma_{0,1}), t(\gamma_{0,2}), t(\gamma_{0,3})|_{\mathcal{M}} = 14,$$

where one can choose $\gamma_{0,j}$ to be S_1 if such merged subpath does not exist on ϕ_j . Now set

$$\mathbb{L}_1 = \mathbb{R}^{(0)}|t(\gamma_{0,1}), t(\gamma_{0,2}), t(\gamma_{0,3})|,$$

and

$$\mathbb{S}^{(1)} = \mathbb{S}^{(0)} \cup \mathbb{L}_1, \quad \mathbb{R}^{(1)} = G \setminus \mathbb{S}^{(1)}.$$

Suppose that we already obtain

$$\mathbb{L}_i = \mathbb{R}^{(i-1)}|t(\gamma_{i-1,1}), t(\gamma_{i-1,2}), t(\gamma_{i-1,3})|,$$

and

$$\mathbb{S}^{(i)} = \mathbb{S}^{(i-1)} \cup \mathbb{L}_i, \quad \mathbb{R}^{(i)} = G \setminus \mathbb{S}^{(i)},$$

where \mathbb{L}_i contains exactly 14 mergings and at least two ψ -terminal merged subpaths. We then continue to pick merged subpath $\gamma_{i,j}$ on ϕ_j from $\mathbb{R}^{(i)}$ such that

$$|\mathbb{R}^{(i)}|t(\gamma_{i,1}), t(\gamma_{i,2}), t(\gamma_{i,3})|_{\mathcal{M}} = 14$$

and there are at least two ψ -terminal mergings in $\mathbb{R}^{(i)}|t(\gamma_{i,1}), t(\gamma_{i,2}), t(\gamma_{i,3})|$. If such $\gamma_{i,j}$'s exist, set

$$\mathbb{L}_{i+1} = \mathbb{R}^{(i)}|t(\gamma_{i,1}), t(\gamma_{i,2}), t(\gamma_{i,3})|,$$

and if $|\mathbb{R}^{(i)}| < 14$, set $\mathbb{L}_{i+1} = \mathbb{R}^{(i)}$ and terminate the iterative procedure. So far, for any obtained "block" \mathbb{L}_{i+1} , either we have $|\mathbb{L}_{i+1}|_{\mathcal{M}} < 14$ or ($|\mathbb{L}_{i+1}|_{\mathcal{M}} = 14$ and there are at least two ψ -terminal mergings in \mathbb{L}_{i+1}); such block \mathbb{L}_{i+1} is said to be *normal*. If $|\mathbb{R}^{(i)}| \geq 14$, however, we cannot find a normal block, we continue the procedure and define a *singular* \mathbb{L}_{i+1} in the following.

Note that $\mathbb{S}^{(i)} = \mathbb{L}_1 \cup \mathbb{L}_2 \cup \dots \cup \mathbb{L}_i$. Let $z_i = \sum_{j=1}^i (x_j - y_j)$, where x_i and y_i denote the number of x -terminal and y -terminal mergings in the ψ -paths in \mathbb{L}_i , respectively; then z_i

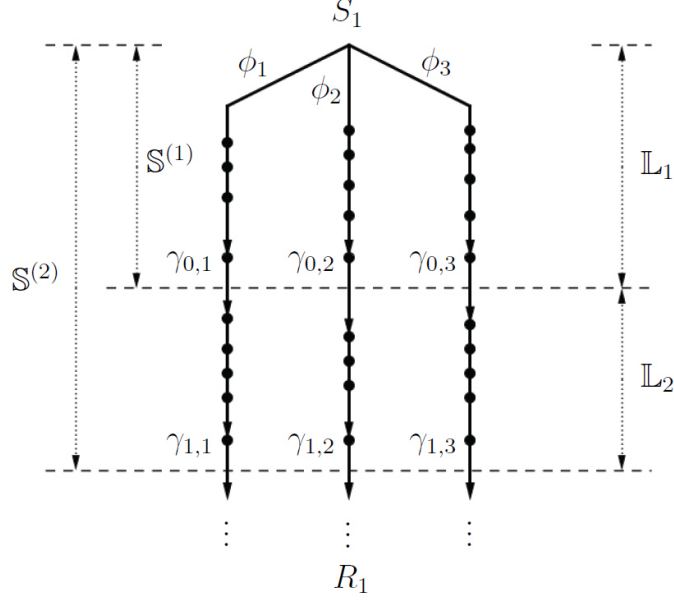


Figure 17: Partition a $(3, n)$ -graph G into blocks

is the number of ψ -paths which can continue to merge within $\mathbb{R}^{(i)}$. If a normal block does not exist after i iterations, necessarily we will have $z_i \geq 3$ (suppose $z_i \leq 2$, by the fact that $\mathcal{M}(3, 3) = 13$ (see Theorem 3.6), we would be able to obtain a normal block \mathbb{L}_{i+1} , which contains two x -terminals or (an x -terminal and a y -terminal)). We say a merged subpath is *critical* within a subgraph of G if the corresponding ψ -path, after merging at this merged subpath, does not merge anymore within this subgraph. It then follows that the number of the critical merged subpaths within $\mathbb{S}^{(i)}$ is z_i .

Now, let \mathbb{K}_i denote the set of all the merged subpaths within $\mathbb{R}^{(i)}$, each of which can semi-reach the tail of some critical merged subpath within $\mathbb{S}^{(i)}$ against ψ . One checks at least one of those ϕ -paths, each of which contains at least one critical merged subpath within $\mathbb{S}^{(i)}$, does not contain any merged subpath within \mathbb{K}_i . Without loss of generality, we assume that $\phi_3 \cap \mathbb{K}_i = \emptyset$. Now we consider the following two cases:

Case 1: $\phi_1 \cap \mathbb{K}_i \neq \emptyset$ and $\phi_2 \cap \mathbb{K}_i \neq \emptyset$. As shown in Figure 18, assume that within \mathbb{K}_i , $\lambda_{i,1}, \lambda_{i,2}$ are the largest merged subpaths on ϕ_1, ϕ_2 , respectively. Now, set

$$\mathbb{L}_{i+1} = \mathbb{R}^{(i)}[t(\lambda_{i,1}), t(\lambda_{i,2})], \quad \mathbb{Q}_i = \phi_1[t(\gamma_{i-1,1}), t(\lambda_{i,1})] \cup \phi_2[t(\gamma_{i-1,2}), t(\lambda_{i,2})].$$

Note that for $\lambda_{i,j}$, $j = 1, 2$, the associated ψ -path, from $\lambda_{i,j}$, may merge outside \mathbb{Q}_i next time. If this ψ -path merges within \mathbb{Q}_i again after a number of mergings outside \mathbb{Q}_i , we call it an *excursive* ψ -path. One checks that there are at most one excursive ψ -path (since, otherwise, we can find a cycle in G , which is a contradiction). On the other hand, for any merged subpath from \mathbb{K}_i other than $\lambda_{i,1}, \lambda_{i,2}$, say μ , the associated ψ -path, from μ , can only merge within \mathbb{Q}_i and will not merge outside \mathbb{Q}_i . So, the number of **connected** ψ -paths that contain at least one merged subpath within $\mathbb{L}_{i+1} \cap \mathbb{Q}_i$ is upper bounded by $y_{i+1} + 2$. Then, by the fact that $\mathcal{M}(2, n) = 3n - 1$ (see Theorem 3.1), we have

$$|\mathbb{L}_{i+1} \cap \mathbb{Q}_i|_{\mathcal{M}} \leq 3(y_{i+1} + 2) - 1. \quad (17)$$

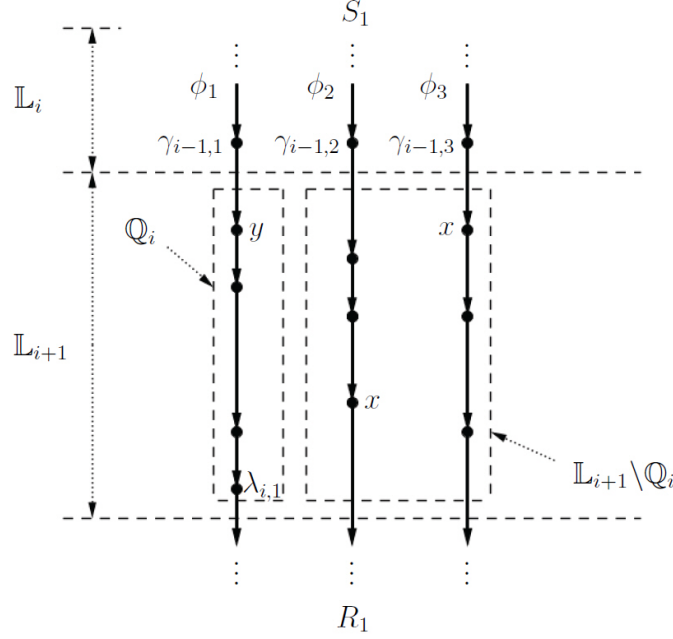


Figure 19: Case 2

that $\mathcal{M}(1, n) = n$ (see Example 2.15 of [5]) and $\mathcal{M}(2, n) = 3n - 1$, we have

$$|\mathbb{L}_{i+1} \cap \mathbb{Q}_i|_{\mathcal{M}} \leq y_{i+1} + 1, \quad |\mathbb{L}_{i+1} \setminus \mathbb{Q}_i|_{\mathcal{M}} \leq 3x_{i+1} - 1.$$

It then immediately follows that $|\mathbb{L}_{i+1}|_{\mathcal{M}} \leq 3x_{i+1} + y_{i+1}$.

Similarly as before, we claim that $x_{i+1} + y_{i+1} \geq 3$. To see this, suppose, by contradiction, that $x_{i+1} + y_{i+1} \leq 2$. From $y_{i+1} + 1 \geq z_i \geq 3$, we infer that $y_{i+1} = 2$ and $x_{i+1} = 0$, and further

$$|\mathbb{L}_{i+1}|_{\mathcal{M}} \leq 3x_{i+1} + y_{i+1} = 2,$$

which implies that we can in fact obtain a normal block, a contradiction.

Combining the above two cases, we conclude that the number of merged subpaths within the singular block \mathbb{L}_{i+1} is upper bounded by $3(x_{i+1} + y_{i+1}) + 7$, where $x_{i+1} + y_{i+1} \geq 3$.

We continue these operations in an iterative fashion to further obtain normal blocks and singular blocks until there are no merged subpaths left in the graph. Suppose there are n_1 singular blocks $\mathbb{L}_{j_1}, \mathbb{L}_{j_2}, \dots, \mathbb{L}_{j_{n_1}}$ and n_2 normal blocks. Note that each singular block has at least three ψ -terminal merged subpaths and each normal block except the last one has at least two ψ -terminal merged subpaths. If the last normal block has at least two ψ -terminal merged subpaths, we then have

$$3n_1 \leq \sum_{i=1}^{n_1} (x_{j_i} + y_{j_i}) \leq 2n - 2n_2.$$

It then follows that

$$|G|_{\mathcal{M}} \leq 14n_2 + \sum_{i=1}^{n_1} [3(x_{j_i} + y_{j_i}) + 7] \leq 14n_2 + 7n_1 + 3(2n - 2n_2) = 6n + 8n_2 + 7n_1 \leq 14n. \quad (19)$$

If the last normal block has only one ψ -terminal merged subpath, necessarily, there are at most three mergings in the last normal block, we then have

$$3n_1 \leq \sum_{i=1}^{n_1} (x_{j_i} + y_{j_i}) \leq 2n - 2(n_2 - 1) - 1.$$

It then follows that

$$|G|_{\mathcal{M}} \leq 14(n_2 - 1) + 3 + \sum_{i=1}^{n_1} [3(x_{j_i} + y_{j_i}) + 7] \leq 6n + 8n_2 + 7n_1 - 8 \leq 14n. \quad (20)$$

Combining (19) and (20), we then have established the theorem. \square

5 Inequalities

Consider two non-reroutable (n, n) -graph $G^{(1)}, G^{(2)}$. For $j = 1, 2$, assume that $G^{(j)}$ has one source $S^{(j)}$, two sinks $R_1^{(j)}, R_2^{(j)}$. Let $\phi^{(j)} = \{\phi_1^{(j)}, \phi_2^{(j)}, \dots, \phi_n^{(j)}\}$ denote the set of Menger's paths from $S^{(j)}$ to $R_1^{(j)}$ and $\psi^{(j)} = \{\psi_1^{(j)}, \psi_2^{(j)}, \dots, \psi_n^{(j)}\}$ denote the set of Menger's paths from $S^{(j)}$ to $R_2^{(j)}$. As before, we assume that, for $1 \leq i \leq n$, paths $\phi_i^{(j)}$ and $\psi_i^{(j)}$ share a starting subpath.

Now, consider the following procedure of concatenating graphs $G^{(1)}$ and $G^{(2)}$:

1. reverse the direction of each edge in $G^{(2)}$ to obtain a new graph $\widehat{G}^{(2)}$ (for $1 \leq i \leq n$, path $\phi_i^{(2)}$ in $G^{(2)}$ becomes path $\hat{\phi}_i^{(2)}$ in $\widehat{G}^{(2)}$ and path $\psi_i^{(2)}$ in $G^{(2)}$ becomes path $\hat{\psi}_i^{(2)}$ in $\widehat{G}^{(2)}$);
2. split $S^{(1)}$ into n copies $S_1^{(1)}, S_2^{(1)}, \dots, S_n^{(1)}$ in $G^{(1)}$ such that paths $\phi_i^{(1)}$ and $\psi_i^{(1)}$ have the same starting point $S_i^{(1)}$; split $S^{(2)}$ into n copies $S_1^{(2)}, S_2^{(2)}, \dots, S_n^{(2)}$ in $\widehat{G}^{(2)}$ such that paths $\hat{\phi}_i^{(2)}$ and $\hat{\psi}_i^{(2)}$ have the same ending point $S_i^{(2)}$;
3. for $1 \leq i \leq n$, identify $S_i^{(1)}$ and $S_i^{(2)}$.

Obviously, such procedure produces an (n, n) -graph with two distinct sources $R_1^{(2)}, R_2^{(2)}$, two sinks $R_1^{(1)}, R_2^{(1)}$, a set of Menger's paths $\{\hat{\phi}_1^{(2)} \circ \phi_1^{(1)}, \hat{\phi}_2^{(2)} \circ \phi_2^{(1)}, \dots, \hat{\phi}_n^{(2)} \circ \phi_n^{(1)}\}$ from $R_1^{(2)}$ to $R_1^{(1)}$ and a set of Menger's paths $\{\hat{\psi}_1^{(2)} \circ \psi_1^{(1)}, \hat{\psi}_2^{(2)} \circ \psi_2^{(1)}, \dots, \hat{\psi}_n^{(2)} \circ \psi_n^{(1)}\}$ from $R_2^{(2)}$ to $R_2^{(1)}$. See Figure 20 for an example where we concatenate two $(3, 3)$ -graphs.

We then have the following lemma.

Lemma 5.1. *The concatenated graph as above is a non-reroutable (n, n) -graph with $|G^{(1)}|_{\mathcal{M}} + |G^{(2)}|_{\mathcal{M}} + n$ mergings.*

The following theorem then immediately follows.

Theorem 5.2.

$$\mathcal{M}(n, n) \geq 2\mathcal{M}^*(n, n) + n.$$

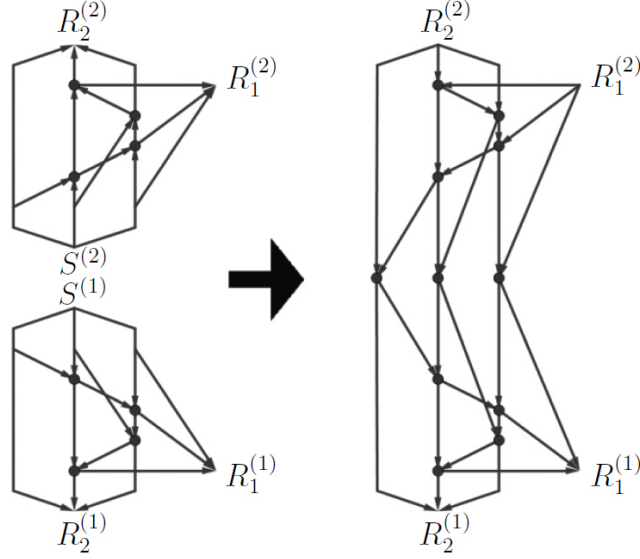


Figure 20: Concatenation of two $(3, 3)$ -graphs

Consider a non-reroutable $(n + 1, n + 1)$ -graph $G^{(1)}$ and a non-reroutable $(n - 1, n - 1)$ -graph $G^{(2)}$. The graph $G^{(1)}$ has one source $S^{(1)}$, two sinks $R_1^{(1)}, R_2^{(1)}$, a set of Menger's paths $\phi = \{\phi_0, \phi_1, \dots, \phi_n\}$ from $S^{(1)}$ to $R_1^{(1)}$ and a set of Menger's paths $\psi = \{\psi_0, \psi_1, \dots, \psi_n\}$ from $S^{(1)}$ to $R_2^{(1)}$. As discussed in Section 2.2, we assume paths ϕ_i and ψ_i share a starting subpath ω_i , and paths ϕ_n, ψ_0 do not merge with any other paths in $G^{(1)}$, directly flowing to the sinks. The graph $G^{(2)}$ has one source $S^{(2)}$, two sinks $R_1^{(2)}, R_2^{(2)}$, a set of Menger's paths $\xi = \{\xi_1, \xi_2, \dots, \xi_{n-1}\}$ from $S^{(2)}$ to $R_1^{(2)}$ and a set of Menger's paths $\eta = \{\eta_1, \eta_2, \dots, \eta_{n-1}\}$ from $S^{(2)}$ to $R_2^{(2)}$. Again, assume paths ξ_i and η_i share a starting subpath.

Now, we consider the following procedure of concatenating graphs $G^{(1)}$ and $G^{(2)}$:

1. reverse the direction of each edge in $G^{(2)}$ to obtain a new graph $\widehat{G}^{(2)}$ (for $1 \leq i \leq n - 1$, path ξ_i in $G^{(2)}$ becomes path $\hat{\xi}_i$ in $\widehat{G}^{(2)}$ and path η_i in $G^{(2)}$ becomes path $\hat{\eta}_i$ in $\widehat{G}^{(2)}$);
2. split $S^{(1)}$ into $n + 1$ copies $S_0^{(1)}, S_1^{(1)}, \dots, S_n^{(1)}$ in $G^{(1)}$ such that paths ϕ_i and ψ_i have the same starting point $S_i^{(1)}$; split $S^{(2)}$ into $n - 1$ copies $S_1^{(2)}, S_2^{(2)}, \dots, S_{n-1}^{(2)}$ in $\widehat{G}^{(2)}$ such that paths $\hat{\xi}_i$ and $\hat{\eta}_i$ have the same ending point $S_i^{(2)}$;
3. delete all edges on ϕ_n , each of which is larger than ω_n ; delete all edges on ψ_0 , each of which is larger than ω_0 ;
4. identify $R_1^{(2)}$ and $S_0^{(1)}$; for $1 \leq i \leq n - 1$, identify $S_i^{(2)}$ and $S_i^{(1)}$; identify $R_2^{(2)}$ and $S_n^{(1)}$.

Obviously, such procedure produces an (n, n) -graph with two distinct sources $R_1^{(2)}, R_2^{(2)}$, two sinks $R_1^{(1)}, R_2^{(1)}$, a set of Menger's paths $\{\phi_0, \hat{\xi}_1 \circ \phi_1, \hat{\xi}_2 \circ \phi_2, \dots, \hat{\xi}_{n-1} \circ \phi_{n-1}, \}$ from $R_1^{(2)}$ to $R_1^{(1)}$ and a set of Menger's paths $\{\hat{\eta}_1 \circ \psi_1, \hat{\eta}_2 \circ \psi_2, \dots, \hat{\eta}_{n-1} \circ \psi_{n-1}, \psi_n\}$ from $R_2^{(2)}$ to $R_2^{(1)}$. For example, in Figure 21, we concatenate a $(2, 2)$ -graph and a $(4, 4)$ -graph to obtain a $(3, 3)$ -graph.

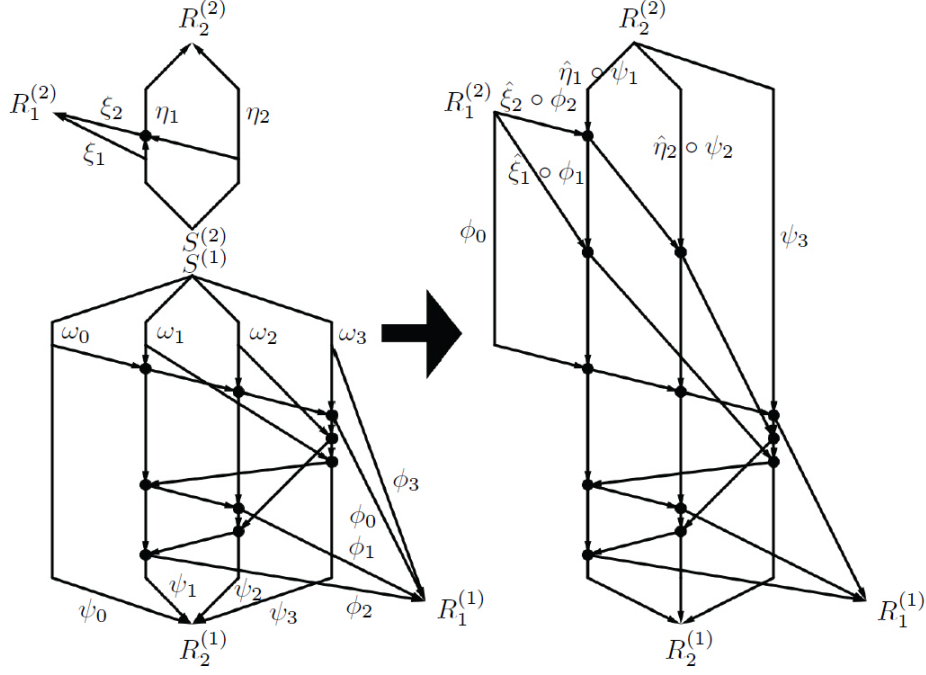


Figure 21: Concatenation of a $(2, 2)$ -graph and a $(4, 4)$ -graph

We then have the following lemma.

Lemma 5.3. *The concatenated graph as above is a non-reroutable (n, n) -graph with $|G^{(1)}|_{\mathcal{M}} + |G^{(2)}|_{\mathcal{M}} + (n - 1)$ mergings.*

It immediately follows that

Theorem 5.4.

$$\mathcal{M}(n, n) \geq \mathcal{M}^*(n + 1, n + 1) + \mathcal{M}^*(n - 1, n - 1) + (n - 1).$$

Consider $n_1 \leq n_2 \leq \dots \leq n_k$. For $j = 1, 2, \dots, k - 1$, consider a non-reroutable (n_j, n_k) -graph $G^{(j)}$ with one source $S^{(j)}$, two sinks $R^{(j)}, \widehat{R}^{(j)}$, a set of Menger's paths $\{\phi_1^{(j)}, \phi_2^{(j)}, \dots, \phi_{n_j}^{(j)}\}$ from $S^{(j)}$ to $R^{(j)}$ and a set of Menger's paths $\{\psi_1^{(j)}, \psi_2^{(j)}, \dots, \psi_{n_k}^{(j)}\}$ from $S^{(j)}$ to $\widehat{R}^{(j)}$. As before, we assume that paths $\phi_i^{(j)}$ and $\psi_i^{(j)}$ share a starting subpath for $1 \leq i \leq n_j$.

Now, consider the following procedure of concatenating graphs $G^{(1)}, G^{(2)}, \dots, G^{(k-1)}$ (see Figure 22 for an example):

1. for $1 \leq j \leq k - 2$, split $\widehat{R}^{(j)}$ into n_k copies $\widehat{R}_1^{(j)}, \widehat{R}_2^{(j)}, \dots, \widehat{R}_{n_k}^{(j)}$ such that path $\psi_i^{(j)}$ has the ending point $\widehat{R}_i^{(j)}$;
2. for $2 \leq j \leq k - 1$, split $S^{(j)}$ into n_k copies $S_1^{(j)}, S_2^{(j)}, \dots, S_{n_k}^{(j)}$ such that paths $\phi_i^{(j)}$ and $\psi_i^{(j)}$ have the same starting point $S_i^{(j)}$;
3. for $1 \leq j \leq k - 2$ and $1 \leq i \leq n_k$, identify $\widehat{R}_i^{(j)}$ and $S_i^{(j+1)}$.

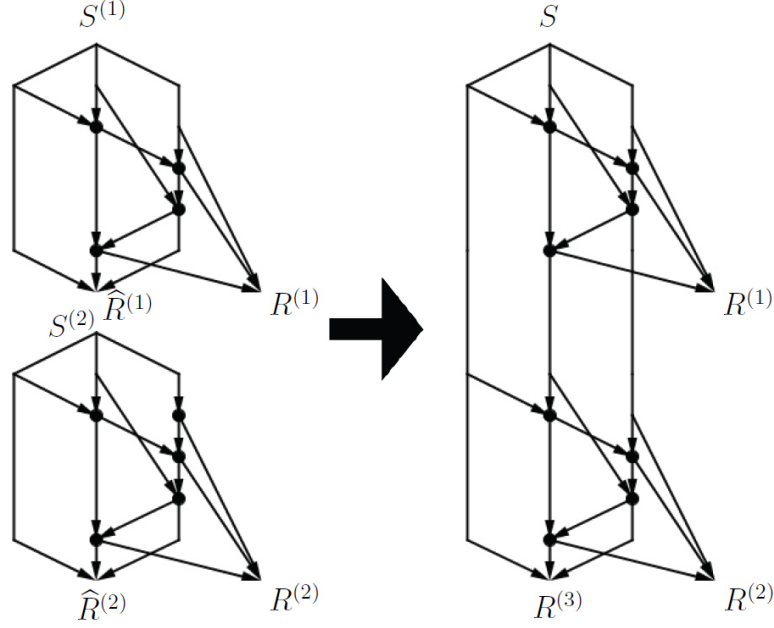


Figure 22: Concatenate two $(3,3)$ -graphs to obtain a $(3,3,3)$ -graph

Relabel $S^{(1)}, \hat{R}^{(k-1)}$ as $S, R^{(k)}$, respectively. We then have an (n_1, n_2, \dots, n_k) -graph with one source S , k sinks $R^{(1)}, R^{(2)}, \dots, R^{(k)}$ and a set of Menger's paths $\{\delta_1^{(j)}, \delta_2^{(j)}, \dots, \delta_{n_j}^{(j)}\}$ from S to $R^{(j)}$ for $1 \leq j \leq k$, where

$$\delta_i^{(j)} = \begin{cases} \psi_i^{(1)} \circ \psi_i^{(2)} \circ \dots \circ \psi_i^{(j-1)} \circ \phi_i^{(j)} & \text{if } 1 \leq j < k, \\ \psi_i^{(1)} \circ \psi_i^{(2)} \circ \dots \circ \psi_i^{(j-1)} \circ \psi_i^{(j)} & \text{if } j = k. \end{cases}$$

We then have the following lemma.

Lemma 5.5. *The concatenated graph as above is a non-reroutable (n_1, n_2, \dots, n_k) -graph with $|G^{(1)}|_{\mathcal{M}} + |G^{(2)}|_{\mathcal{M}} + \dots + |G^{(k-1)}|_{\mathcal{M}}$ mergings.*

It immediately follows that

Theorem 5.6. *For $n_1 \leq n_2 \leq \dots \leq n_k$,*

$$\mathcal{M}^*(n_1, n_2, \dots, n_k) \geq \sum_{i=1}^{k-1} \mathcal{M}^*(n_i, n_i).$$

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